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MATHEMATICS MAGAZINE

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CONTENTS

THE EDITOR'S PAGE

Articulation

The traditional mathematics curriculum in American schools and colleges placed rather definite areas of responsibility upon the high schools and the colleges and placed a rather definite line of demarcation between them. The high schools taught well defined courses in algebra, geometry, and trigonometry. The colleges built upon this foundation and continued with college algebra, analytic geometry, and calculus, thence into advanced mathematics.

Today we are witnessing and participating in a vigorous and extensive reorganization of the entire mathematics curriculum. Well financed commissions, committees and study groups have come forth with broad recommendations concerning the curriculum of both the high schools and the colleges. Sets, inequalities, probability, and statistics are among the new subjects to be introduced into the high school mathematics courses. Sets, linear algebras, probability, structure of numbers, and logic are only a few of the new topics recommended for inclusion in the undergraduate college curriculum.

Herein lies a problem which must be solved if we are to have a well articulated program in the high schools and the colleges. Which of the new topics are to be the responsibility of the high schools and which belong in the colleges. The C.U.P. calls for sets in the Universal Mathematics course in colleges. The Commission of Mathematics calls for sets in high school mathematics courses. Who is to teach what? We need to develop a clear cut division of responsibility between the high schools and the colleges.

Now that a number of commissions and committees have come forth with excellent recommendations at both the secondary and the collegiate levels, we need to articulate the two. Should we not have a Committee on Articulation? The leaders in curriculum revision at both levels should be brought together to insure that careful articulation is an integral part of the final recommendations offered in the revision of both educational levels.

R.E.H.

AN ANALYTIC METHOD FOR THE "DIFFICULT CROSSING" PUZZLES

Benjamin L. Schwartz

1. *Introduction.*

In this paper a family of well-known recreational mathematical problems are attacked by a novel method, involving the use of linear graph theory. An analytical technique is thereby provided to resolve questions which have heretofore been amenable only to trial and error solution. The mathematical recreations involved are those termed "Difficult Crossings" in [1], and a typical example of the problems, paraphrased from that reference is given below.

2. *Example.*

A group consisting of three cannibals and three missionaries seeks to cross a river. A boat is available which will hold up to two people. If the missionaries on either side of the river are outnumbered at any time by the cannibals on that side, even momentarily, the cannibals will do away with the unfortunate, out-numbered missionaries. What schedule of crossings can be devised to permit the entire party to cross safely?

The method we shall use on this problem and others of similar ilk is that of representing it as a linear graph, and applying the results of graph theory. To this end, we herewith introduce some of the applicable graph theoretic concepts and results.

3. *Graph Theory Fundamentals.*

By an *oriented linear graph*, L, we mean an ordered pair $\mathsf{L} = (V, E)$ with the following properties:

(1) V is a finite set. The elements of V are called *vertices* of L.

(2) E is a relation on V. The members of E are called *edges* of L, and if $e = (v_i, v_j) \in E$, we call e the edge *from* v_i *to* v_j.

By a *path* P in L, we mean a sequence of vertices of L,

$$P = v_1, v_2, \cdots, v_k$$

such that if v_i and v_{i+1} are two successive members of P, then (v_i, v_{i+1}) is an edge. We say that P is a path *from* v_1 *to* v_k. We say v_i is *connected* to v_j if there exists a path from v_i to v_j.

The questions now arise, given an oriented linear graph and two selected vertices, v_i and v_k: is v_i connected to v_k, and if so, what is a path from v_i to v_k? In the literature of graph theory, such as [3] and [5] these questions and some of their generalizations are answered. We need

not develop the results of those papers at length here, but a simple explanation will illustrate the procedures used.

4. *Determination of Paths.*

Consider an oriented linear graph $\mathsf{L} = (V, E)$ with n vertices $v_1, \dots, v_n$. Define a $(n \times n)$ matrix C, called the primary connection matrix of L, by the following rule.

$$c_{ij} = 1 \quad \text{if } (v_i, v_j) \in E \ ,$$

$$c_{ij} = 0 \quad \text{otherwise.}$$

We now define a novel matrix "multiplication" as follows:

$$R = P \cdot Q = [p_{ij}] \cdot [q_{ij}] = [r_{ij}] = [\max_{1 \leqq k \leqq n} \min (p_{ik}, q_{kj})] \ .$$

Under this rule, which is used throughout the remainder of this paper, we can now assert the theorem:

v_i is connected to v_j if, and only if, for some p, $c_{ij}^p = 1$, where of course, c_{ij}^p denotes the (i, j) element of C^p.

We briefly sketch the proof. Suppose first, for example, $c_{ij}^2 = 1$. Here $p = 2$. By definition, we have $\max\limits_k \min (c_{ik}, c_{kj}) = 1$, hence, for some k, say k^*, we have $c_{ik^*} = c_{k^*j} = 1$. It then follows that v_i, v_{k^*}, v_j is a 3-vertex path from v_i to v_j. Similarly, for higher values of p, the determination of a k^* such that

$$c_{ik^*} = c_{k^*j}^{p-1} = 1$$

inductively reduces the order of the problem by one level. For any value of p, the steps are reversible, so the converse also holds. Furthermore, the method outlined here actually reveals the path from v_i to v_j, since at each stage (v_i, v_{k^*}) is an edge, and the sequence of edges so determined is the desired path.

5. *Difficult Crossings Viewed as Graphs.*

To apply this technique to the Difficult Crossing problems, one further extension is required, which we shall illustrate by our example of the backsliding cannibals and missionaries.

We first convert the Difficult Crossing problem into graph theoretic language by defining the vertices and edges to be considered. Let the ordered pair (m, c) denote the state of the problem in which m missionaries and c cannibals are on the near side of the river. Since $0 \leqq m \leqq 3$ and $0 \leqq c \leqq 3$, there are 16 possible such pairs. But some of these must be excluded because of the greed of the cannibals. For example, (1, 3) is forbidden. So also is (2, 1), since the lone missionary on the far side will be

doomed. By complete enumeration of these cases, we find that there remain ten admissible pairs of the (m, c) form, namely:

$$\begin{aligned}
v_1 &= (3, 3)\\
v_2 &= (3, 2)\\
v_3 &= (3, 1)\\
v_4 &= (3, 0)\\
v_5 &= (2, 2)\\
v_6 &= (1, 1)\\
v_7 &= (0, 3)\\
v_8 &= (0, 2)\\
v_9 &= (0, 1)\\
v_{10} &= (0, 0)\,.
\end{aligned}$$

These ten ordered pairs will constitute the vertices of our graph. (In general, the procedure is to enumerate all possible combinations of the entities in the problem statement, and eliminate those which are excluded by the conditions of the puzzle. In the final section, we shall illustrate this with another example.)

The problem now takes the form of finding a path from $v_1 = (3, 3)$ to $v_{10} = (0, 0)$.

6. *A Difficulty, and its Resolution.*

In attempting to perform the latter operation, we encounter a dilemma. An edge, permitting transition from one vertex to another, is navigated by means of the boat, whose capacity is two persons. Thus, the directed edge (v_1, v_5), i.e., $(3, 3) \to (2, 2)$, appears possible. And it is, *provided the boat is on the near side of the river.* If the boat is across the river, the directed edge is (v_5, v_1). How shall we then define the edges of the graph?

We meet this dilemma in two steps. First, define a set E of edges assuming the boat always on the near side. The primary connection matrix C of the graph so generated is:

$$C = \begin{bmatrix}
0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}$$

We now observe that the transpose of this matrix, C^T, displays those edges which can be navigated when the boat is on the far side.

The second step is the recognition that the boat must alternate sides. Hence, the matrix $(C \cdot C^T)$ represents the journeys attainable in one round trip, starting with the boat on the near side.

To see this, consider a typical element in the (i, j) position of $(C \cdot C^T)$. This element will be a 1 if, and only if,

$$\max_{1 \leq k \leq n} \min (c_{ik}, c_{kj}^T) \quad \text{is equal to 1.}$$

This holds if, and only if, for some k, say k^*, we have $c_{ik^*} = c_{k^*j}^T = 1$. But this implies that (v_i, v_{k^*}) and (v_{k^*}, v_j) are edges which can be traversed on trips over and back, respectively.

In similar manner, it is easy to see that after p round trips, the matrix of available journeys is $(C \cdot C^T)^p$. If the p found trips are followed by one final trip to the far bank, the matrix becomes $(C \cdot C^T)^p \cdot C$.

7. *Algebraic Formulation and Solution.*

The entire problem of the crossings now becomes the purely algebraic question of the least power p in $(C \cdot C^T)^p \cdot C$ which causes the (1, 10) element in the matrix product to assume the value 1. This can be solved in tedious but direct manner by explicit computation of $(C \cdot C^T)^p \cdot C$ for $p = 0$, 1, 2, etc. Once the power has been ascertained, the actual reconstruction of the steps in the multiplication which led to it can be performed by a finite induction method similar to that illustrated above in Section 4. To do this in the present case with the matrix C of Section 5 is quite lengthy, but perfectly straightforward. The results are herewith given without reproducing the calculations in detail. Five round trips are required, plus the final single crossing. Essentially, only one sequence of vertices can be followed, except that an option of two choices exists at the second and eleventh stage. The sequence of vertices representing the solution is 1, 3 or 5, 2, 4, 3, 6, 5, 8, 7, 9, 6 or 8, 10. In words,

1a) Two cannibals cross and one returns,

or

1b) A cannibal and a missionary cross and the missionary returns.

2) Two cannibals cross and one returns.

3) Two missionaries cross; a missionary and a cannibal return.

4) Two missionaries cross, a cannibal returns.

5) Two cannibals cross.

6) Either a missionary or a cannibal returns and picks up the other cannibal, and they cross together.

8. *Generalization.*

While the generalization to other Difficult Crossings is clear, the reader may now ask what will happen if a proposed puzzle has no solution.

The answer is that the algebraic approach will reveal this fact and provide a rigorous proof of impossibility. To see this, notice that if $k < k'$, then $(C \cdot C^T)^k \leqq (C \cdot C^T)^{k'}$ where the matrix dominance relation $\leqq$ here means term-by-term dominance. Since each matrix consists exclusively of 0 and 1 entries, all we have to prove to verify this assertion is that if the (i, j) element of $(C \cdot C^T)^k$ is a 1, then so also is the (i, j) element of $(C \cdot C^T)^{k'}$. In terms of crossings, this means that if the condition represented by vertex v_j can be achieved in k round trips of the boat, starting from the condition representing vertex v_i, then it can also be achieved in k' round trips. But this is trivial since the last $(k'-k)$ round trips can all be made with the same passenger list. (The purist will find that it is not hard to translate this heuristic argument into a rigorous algebraic one.) Hence, the sequence $(C \cdot C^T)^k$ $(k = 1, 2, 3, \cdots)$, is monotone increasing, bounded above (by the matrix of all 1's), and therefore, has a largest member, say D. We can identify D when we reach it in the sequence by the fact that $D \cdot (C \cdot C^T) = D$. Now if the appropriate element of $D \cdot C$ is 1, the proposed puzzle is possible; otherwise, impossible. For example, the reader is invited to verify that the cannibal-missionary puzzle with four of each type cannot be solved.

9. *Additional Example*.

We conclude with an example along the same theme which illustrates how additional conditions may be accommodated. Suppose again our population consists of three missionaries and three cannibals, but let us assume only one of the cannibals is able to row the boat, although all the missionaries are. The boat again holds two passengers, and the missionaries may not be out-numbered. As before, we formulate this in graph theoretic language by defining the vertices. The population is now considered to consist of three types: missionaries, rowing cannibals, non-rowing cannibals. An ordered triple (m, r, c) can be used to represent the number of each on the near side of the river. We have the conditions:

$$0 \leqq m \leqq 3 \qquad 0 \leqq r \leqq 1 \qquad 0 \leqq c \leqq 2 .$$

There are, therefore, 24 possibilities, but as before, some of these can be excluded by the conditions of the problem. Some 16 remain as the vertices of our graph, viz:

$$v_1 = (3, 1, 2)$$

$$v_2 = (3, 0, 2)$$

$$v_3 = (3, 1, 1)$$

$$v_4 = (3, 1, 0)$$

$$v_5 = (3, 0, 1)$$

$$v_6 = (3, 0, 0)$$

$$v_7 = (2, 0, 2)$$
$$v_8 = (2, 1, 1)$$
$$v_9 = (1, 0, 1)$$
$$v_{10} = (1, 1, 0)$$
$$v_{11} = (0, 1, 2)$$
$$v_{12} = (0, 1, 1)$$
$$v_{13} = (0, 0, 2)$$
$$v_{14} = (0, 0, 1)$$
$$v_{15} = (0, 1, 0)$$
$$v_{16} = (0, 0, 0)\,.$$

The primary connection matrix, assuming as before the boat on the near side, now turns out to be

$$\begin{bmatrix}
0&1&0&0&1&0&1&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&1&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&0&1&0&0&0\\
0&0&0&0&0&0&0&0&1&1&0&1&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&0&0&1&0&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1\\
0&0&0&0&0&0&0&0&0&0&0&0&1&1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0
\end{bmatrix}$$

From this point on, the procedure parallels exactly that described previously. The solution is obtained with $p = 6$, i. e., six round trips, plus one additional final crossing. The 13 separate one-way trips take us through the following path of vertices: 1, 5 or 8, 3, 6, 4. 10, 8, 9, 7, 13, 11, 14, 9 or 12, 16. And again, in words:

1) Either the rowing cannibal, or a missionary, takes one of the other cannibals across, leaves him, and returns.

2) Two cannibals cross, and a cannibal returns.

3) Two missionaries cross, a missionary and a cannibal return.

4) A missionary crosses with the rowing cannibal, leaves him on the far side, picks up the other cannibal, and returns.

5) Two missionaries cross.

6) The rowing cannibal returns, picks up one of the other cannibals,

and recrosses.

7) Either a missionary, or the rowing cannibal, crosses to pick up the last cannibal, and returns.

10. *Concluding Remarks.*

The reader who has performed the calculations indicated in our various examples will have discovered that the labor involved in the matrix operations is considerable. We concede this at once. It is true that certain short cuts reveal themselves as the calculations proceed. For example, the property of having a zero row or column in the $(C \cdot C^T)$ matrix is hereditary. In the example of the preceeding section, this fact effectively reduces the problem from a 16×16 matrix computation to a 12×12. Nevertheless, the amount of work remains large. However, to our knowledge, this technique, laborious as it may be, is the only one which provides an analytical procedure for attacking the Difficult Crossing type of problem in contrast to the trial-and-error methods elsewhere used.

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DETERMINANTS FOR 1961

C. W. Trigg

1)
$$\begin{vmatrix} 1 & 9 & 6 & 1 \\ 9 & 0 & 0 & 6 \\ 6 & 0 & 0 & 9 \\ 1 & 6 & 9 & 1 \end{vmatrix} = (45)^2 = - \begin{vmatrix} 1 & 9 & 6 & 1 \\ 6 & 0 & 0 & 9 \\ 9 & 0 & 0 & 6 \\ 1 & 6 & 9 & 1 \end{vmatrix} .$$

2)
$$\begin{vmatrix} 1 & 9 & 6 & 1 \\ 9 & 1 & 1 & 6 \\ 6 & 1 & 1 & 9 \\ 1 & 6 & 9 & 1 \end{vmatrix} = 3^2(15^2 - 2^2) = - \begin{vmatrix} 1 & 9 & 6 & 1 \\ 6 & 1 & 1 & 9 \\ 9 & 1 & 1 & 6 \\ 1 & 6 & 9 & 1 \end{vmatrix} .$$

3) The absolute values of the 12 second order determinants formed from the digits of 1961 are equal in groups of four and of eight. Representative samples are, respectively:

$$\begin{vmatrix} 1 & 9 \\ 6 & 1 \end{vmatrix} = -53 \quad \text{and} \quad \begin{vmatrix} 1 & 9 \\ 1 & 6 \end{vmatrix} = -3 .$$

4)
$$\begin{vmatrix} 1 & 9 & 6 \\ 9 & 6 & 1 \\ 6 & 1 & 9 \end{vmatrix} = -(28)^2 , \quad \text{and} \quad \begin{vmatrix} 1 & 9 & 6 \\ 9 & 6 & 1 \\ 1 & 6 & 9 \end{vmatrix} = 6 \cdot 7 \cdot 8 .$$

5) The circulant
$$\begin{vmatrix} 1 & 9 & 6 & 1 \\ 9 & 6 & 1 & 1 \\ 6 & 1 & 1 & 9 \\ 1 & 1 & 9 & 6 \end{vmatrix}$$

is divisible by
$$\begin{vmatrix} 1 & 9 \\ 1 & 6 \end{vmatrix} .$$

That is, $4539/(-3) = -1513 = -17 \cdot 89$.

6)
$$\begin{vmatrix} 1 & 9 \\ 6 & 1 \end{vmatrix} \text{ divides } 1961 ,$$

thus $1961/(-53) = -37$.

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THE PROJECTION OF A VECTOR ON A PLANE

H. Randolph Pyle

Introduction. The usual method of finding the projection of a vector on a plane in 3-space is to find the projection of the vector on the normal to the plane and subtract this projection from the original vector. This method does not lend itself to projections on planes in spaces of higher dimensions because there is no unique normal to the plane in such spaces.

An alternative method which can be readily generalized is to find the line in the plane which makes a minimum angle with the given vector, and to define the projection of the vector on the plane as the projection of the given vector on this line. This can be done by using two non-parallel vectors in the plane and their reciprocals. The vector projection can be simply expressed in terms of these vectors.

Reciprocal sets of vectors. Let us assume that we have an n-space in which the cosine of the angle between two vectors may be expressed in terms of their inner product. These vectors will have n coordinates. If the plane is spanned by the vectors A_1 and A_2, their reciprocals $\bar{A}_1$ and $\bar{A}_2$ can be found by matrix methods. The plane may be represented by the $2 \times n$ matrix $A = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}$. If $A^t = (A_1^t, A_2^t)$, where t indicates the transpose, the matrix

$$B = AA^t = \begin{pmatrix} A_1 \cdot A_1 & A_1 \cdot A_2 \\ A_2 \cdot A_1 & A_2 \cdot A_2 \end{pmatrix}$$

is square and non-singular, and so has an inverse. If the reciprocal set of vectors is represented by $\bar{A} = \begin{pmatrix} \bar{A}_1 \\ \bar{A}_2 \end{pmatrix}$, then $\bar{A} = B^{-1}A$.

Let

$$B^{-1} = \begin{pmatrix} \alpha_{11} & \alpha_{21} \\ \alpha_{12} & \alpha_{22} \end{pmatrix}, \quad (\alpha_{ij} = \alpha_{ji}),$$

where α_{ij} is the reduced cofactor of $A_i \cdot A_j$ in B. These cofactors are simply expressed for a 2×2 matrix, but we use them in order to make generalization easier.

The reciprocal vectors are

$$\bar{A}_1 = \alpha_{11}A_1 + \alpha_{21}A_2, \quad \bar{A}_2 = \alpha_{12}A_1 + \alpha_{22}A_2 .$$

This is easily verified if we remember that two sets of vectors are reciprocal

if $A_i \cdot \overline{A}_j = \delta_{ij}$, where δ_{ij} is the Kronecker delta.

The projection of a vector on the plane. We shall use the method of Forsyth [1] to prove that if U is any vector in the n-space, the projection

$$U_p = (U \cdot A_1)\overline{A}_1 + (U \cdot A_2)\overline{A}_2 = (U \cdot \overline{A}_1)A_1 + (U \cdot \overline{A}_2)A_2 ,$$

and that it may be simply expressed in terms of A_1 and A_2.

Any vector V in the plane may be written

$$V = \lambda_1 A_1 + \lambda_2 A_2 = \mu_1 \overline{A}_1 + \mu_2 \overline{A}_2 .$$

We are interested in finding the vector V such that the angle between U and V is a minimum. Let $v = |V|$, $u = |U|$, and θ be the angle between U and V. Then

$$V \cdot U = vu \cos\theta = \lambda_1(A_1 \cdot U) + \lambda_2(A_2 \cdot U) .$$

There will always be λ's which make V perpendicular to U, so that θ is a right angle and $\cos^2\theta$ has the minimum value zero. When U is perpendicular to the plane, the λ's are indeterminate. When this is not the case, $\cos^2\theta$ will never be larger than one and for any U there must be values of the λ's which give a maximum value of $\cos^2\theta$ and hence a minimum value of θ. Since

$$V = \lambda_1 A_1 + \lambda_2 A_2 , \quad \frac{\partial V}{\partial \lambda_i} = A_i .$$

Also since $v^2 = V \cdot V$,

$$\frac{v \partial v}{\partial \lambda_i} = V \cdot \frac{\partial V}{\partial \lambda_i} = V \cdot A_i .$$

U does not contain the λ's, so that

$$\frac{\partial(V \cdot U)}{\partial \lambda_i} = \frac{\partial V}{\partial \lambda_i} \cdot U = A_i \cdot U .$$

Since $\cos\theta = \dfrac{V \cdot U}{vu}$,

$$\frac{\partial(\cos\theta)}{\partial \lambda_i} = \frac{1}{uv^2}\left[\frac{v\partial(V \cdot U)}{\partial \lambda_i} - (V \cdot U)\frac{\partial v}{\partial \lambda_i}\right]$$

$$= \frac{1}{uv^3}\left[\frac{v^2\partial(V \cdot U)}{\partial \lambda_i} - (V \cdot U)\frac{v\partial v}{\partial \lambda_i}\right]$$

$$= \frac{1}{uv^3}\left[v^2(U \cdot A_i) - (V \cdot U)(V \cdot A_i)\right] .$$

This is zero for maximum $\cos\theta$ and we have

$$V \cdot A_i = \frac{v^2(U \cdot A_i)}{V \cdot U},$$

where v^2 and $V \cdot U$ represent the values that make $\cos\theta$ a maximum. Let

$$\frac{v^2}{V \cdot U} = b .$$

Then

$$V \cdot A_i = (\mu_1 \bar{A}_1 + \mu_2 \bar{A}_2) \cdot A_i = b(U \cdot A_i) .$$

Since $\bar{A}_i \cdot A_j = \delta_{ij}$, we have

$$\mu_1 = b(U \cdot A_1) , \quad \mu_2 = b(U \cdot A_2) , \quad \text{and}$$

$$V = b[(U \cdot A_1)\bar{A}_1 + (U \cdot A_2)\bar{A}_2] .$$

If U_p is the projection of U on the plane,

$$U_p = (u \cos\theta)\frac{V}{v} = \left(\frac{V \cdot U}{v^2}\right) V = b^{-1}V ,$$

so that

$$U_p = (U \cdot A_1)\bar{A}_1 + (U \cdot A_2)\bar{A}_2 .$$

By interchanging A_i and $\bar{A}_i$, we have

$$U_p = (U \cdot \bar{A}_1)A_1 + (U \cdot \bar{A}_2)A_2 .$$

Multiplying the two different forms of U_p gives

$$U_p \cdot U_p = (U \cdot A_1)(U \cdot \bar{A}_1) + (U \cdot A_2)(U \cdot \bar{A}_2) = U \cdot U_p$$

$$\text{or} \quad |U_p|^2 = |U||U_p| \cos\theta \qquad \text{and} \quad |U_p| = |U| \cos\theta .$$

Substituting $\bar{A}_i = \alpha_{1i} A_1 + \alpha_{2i} A_2$ in $U_p = (U \cdot A_1)\bar{A}_1 + (U \cdot A_2)\bar{A}_2$ gives

$$U_p = [(U \cdot A_1)\alpha_{11} + (U \cdot A_2)\alpha_{12}]A_1 + [(U \cdot A_1)\alpha_{21} + (U \cdot A_2)\alpha_{22}]A_2 .$$

$$U_p = \frac{1}{|B|}\left[\begin{vmatrix} U \cdot A_1 & U \cdot A_2 \\ A_2 \cdot A_1 & A_2 \cdot A_2 \end{vmatrix} A_1 + \begin{vmatrix} A_1 \cdot A_1 & A_1 \cdot A_2 \\ U \cdot A_1 & U \cdot A_2 \end{vmatrix} A_2\right] .$$

Projection of a vector on an m-flat. In order to find the projection of a vector on an m-flat in n-space ($m \leq n$), it is necessary to increase the number of vectors A_i from two to m. This does not change the method of proof and the theorems generalize in an obvious manner.

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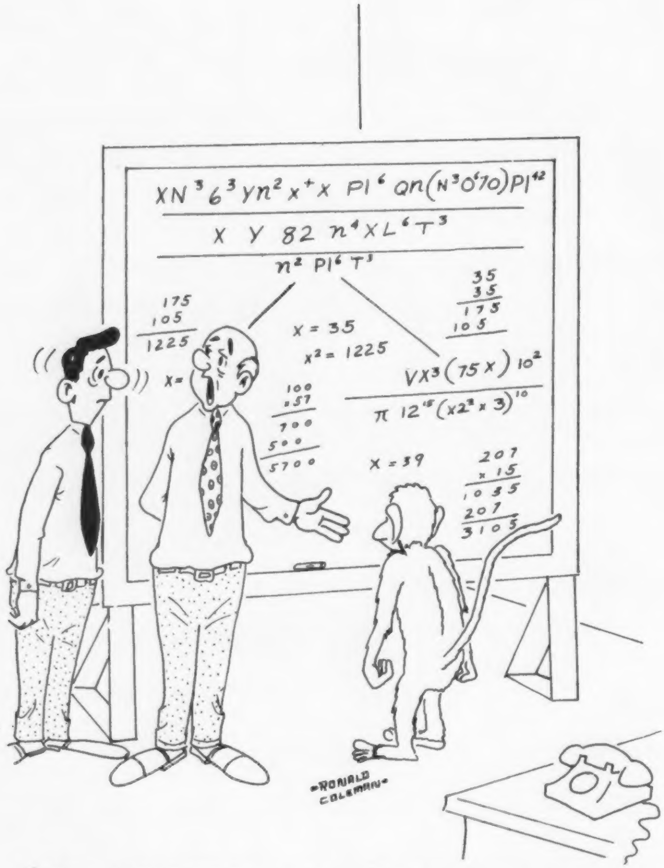
Whittier College



"HE WANTS TO DOUBLE-CHECK OUR FIGURES FOR THAT SATELLITE WE'RE GOING TO LAUNCH!"

ON THE ARZELÀ-ASCOLI THEOREM

J. W. Green and F. A. Valentine*

One of the fundamental tools in analysis is the theorem of C. Arzelà-J. Ascoli, which in one form states that a sequence S_0 of uniformly bounded, uniformly equicontinuous functions on a closed and bounded interval has a uniformly convergent subsequence. Its applications to existence theorems in differential and integral equations, conformal mapping and extremal problems in complex variable theory, the calculus of variations, and other branches of analysis are many and well-known, and proofs of the theorem are to be found in most of the standard textbooks dealing with these subjects.

These proofs generally proceed as follows: First, a countable dense set of points $x_1, x_2, x_3, \cdots$, of the interval is selected. By the Bolzano-Weierstrass theorem, there is a subsequence S_1 of S_0 which converges at x_1. Similarly, there is a subsequence S_2 of S_1 which converges at x_2, etc. A diagonal process supplies a subsequence S of S_0 which converges at all x_i, and use of the uniform equicontinuity shows that S converges uniformly throughout the interval.

This is a very admirable proof – short, easy to follow, easy to remember, and making use of no more powerful tools than are necessary. In short, an elegant proof. Nevertheless, inexperienced students still have some difficulty in really understanding what is going on and why the proof works. For one thing, the appeal to something as impressive sounding as the Bolzano-Weierstrass theorem so early in the game makes them a little uneasy; for another, the diagonal process looks positively miraculous; and for still another, the end play with the epsilons and deltas, though believable, looks somewhat as though it had been done with mirrors.

In this note we shall give a proof of the Arzelà-Ascoli theorem, which, in the simplest case of real valued functions on a closed interval, can be easily explained to an undergraduate class studying infinite series for the first time. To assist in this explanation we shall even stoop to the use of a figure. Yet, on the other hand, this proof is capable of wide abstract generalization without essential change of method. The basic idea of the proof is that used in Hadwiger's proof of the Blaschke selection theorem in the theory of convex bodies, and amounts to the fact that if infinitely many things are distributed among finitely many boxes, some box must

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have infinitely many things in it. We shall not be able to avoid the diagonal process, but when it comes, its motivation will be obvious.

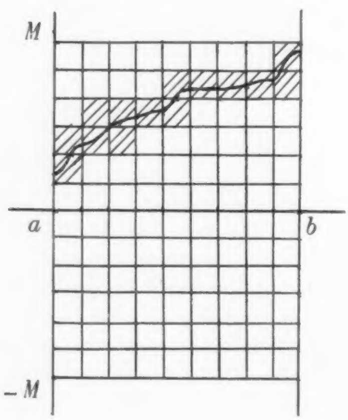
We recall that a collection of functions f is uniformly equicontinuous on an interval if to each positive ϵ there corresponds a positive δ such that $|f(x)-f(y)| < \epsilon$ whenever x, y belong to the interval, $|x-y| < \delta$, and when f is in the collection.

THEOREM. (Arzelà-Ascoli). *Let $S_0 : f_1, f_2, f_3, \cdots,$ be a uniformly bounded uniformly equicontinuous sequence of functions on the closed interval $[a, b]$. There exists a subsequence S of S_0 which converges uniformly on $[a, b]$.*

The proof is contained almost entirely in the following lemma, which we shall prove before going on to the theorem proper.

LEMMA. *Let S_0 satisfy the hypothesis of the theorem. Then given $\epsilon > 0$, there exists a subsequence S of S_0 such that $|f(x)-g(x)| < \epsilon$ for all x in $[a, b]$ and for all f and g in S.*

Proof of lemma. Since the f_i are uniformly bounded, there exists some positive M such that $|f_i(x)| \leqq M$ for all i. Let n be a positive integer such that $M/n < \epsilon/2$. In the x-y plane, draw horizontal lines at $y = 0, \pm M/n, \pm 2M/n, \cdots, \pm M$. Given the number M/n, let δ correspond according to the definition of uniform equicontinuity, and let k be a positive integer such that $(b-a)/k < \delta$. Divide the interval $[a, b]$ into k equal parts and draw vertical lines at the points of subdivision. These horizontal and vertical lines divide the large rectangle $a \leqq x \leqq b$, $-M \leqq y \leqq M$ into exactly $2kn$ small rectangles of dimension $(b-a)/k$ by M/n as shown in the figure



Now the graph of each function $y = f_i(x)$ lies in the large rectangle. Suppose we associate with each f_i the subcollection $R(i)$ of small rectangles (considered closed, for simplicity) through which the graph of $f_i(x)$ passes.

For the f_i in the figure, $R(i)$ consists of the shaded area.

The important thing to note is this: the portion of R_i in any vertical column consists either of one or two adjacent small rectangles. This is because of the fact that at two different values of x differing by no more than $(b-a)/k$ (and hence by less than δ) no $f_i(x)$ can have values differing by as much as M/n.

Since there are only a finite number $2nk$ of small rectangles altogether, there are only a finite number of subcollections thereof (in fact, 2^{2nk}). Consequently, some subcollection, which we shall call R_0, must be associated with infinitely many of the f_i. Let S be the subsequence of S_0 consisting of exactly those f_i associated with R_0. From what has been said about the vertical columns, it is clear that if f and g are any two members of S

$$|f(x)-g(x)| \leqq 2M/N < \epsilon ,$$

and the lemma is proved.

Proof of theorem. Taking $\epsilon = 1/2$, we are assured by the lemma of the existence of a subsequence S_1 of S_0, any two members of which differ numerically by less than 1/2. Similarly if $\epsilon = 1/2^2$, there is a subsequence S_2 of S_1, any two members of which differ by less than $1/2^2$, etc. In this way we obtain the sequences S_0, S_1, S_2, S_3, $\cdots$, each one of which is a subsequence of the previous one. Here is where the diagonal process appears. If we take $g_n(x)$ to be the n-th member of S_n, then the sequence $g_1, g_2, g_3, \cdots$, is a subsequence of S_0 and clearly

$$|g_n(x) - g_m(x)| < 1/2^n$$

if $m > n$. That is, the sequence $g_1, g_2, g_3, \cdots$, satisfies the Cauchy convergence criterion uniformly, and so it is the subsequence whose existence is asserted in the statement of the theorem.

The preceeding proof is quite easily extended to much more general situations. For example, the fact that the range of x was a real interval did not enter essentially into the proof, nor the fact that the values of the functions were real. What did enter in essentially was the triangular inequality, and so it is natural to attempt to generalize the proof to metric spaces. A natural generalization of a closed interval on the line is a compact metric space; however, this is more restrictive than is necessary and something weaker will do. It turns out that the property of the metric space that is needed is the following: Given $\epsilon > 0$ there exists a finite set of points E of the space which is ϵ-dense in the space; that is, has the property that every point of the space is within a distance ϵ of some point of E. Such spaces are called precompact. It follows immediately from the Heine-Borel theorem that compact spaces are precompact.

It is also convenient to use the notion of modulus of continuity of a continuous function. A modulus of continuity of a continuous function f is a function ω of the positive real variable δ such that (i) $\omega(\delta)$ tends to zero as δ tends to zero, and (ii) $|f(x)-f(y)| < \omega(\delta)$ when $|x-y| < \delta$. If x and $f(x)$ are not real, but lie in certain metric spaces, then absolute values are replaced by the appropriate metrics. It is obvious that a collection of functions is uniformly equicontinuous if and only if these functions have a common modulus of continuity.

THEOREM. (Arzela-Ascoli, in more general form). *Let D and R be precompact metric spaces and $S_0 : f_1, f_2, f_3, \cdots$, a uniformly equicontinuous sequence of functions on D to R. There exists a subsequence S of S_0 which is uniformly Cauchy-convergent on D.*

Let ρ and σ be the metrics in D and R respectively. The proof of the theorem will depend on the following lemma which is an obvious generalization of the earlier one.

LEMMA. *Let D, R, and S_0 satisfy the hypothesis of the immediately preceeding theorem. Then given $\epsilon > 0$, there exists a subsequence S of S_0 such that $\sigma(f(x), g(x)) < \epsilon$ for all x in D and all f, g in S.*

Proof of lemma. Let

$$R : y_1, y_2, \cdots, y_n$$

be a finite set of n points of R which is $\epsilon/4$ dense in R. Let δ be such that $\omega(\delta) < \epsilon/4$ where ω is the common modulus of continuity of the members of S_0. Let

$$D : x_1, x_2, \cdots, x_m$$

be a finite set of m points of D which is δ dense in D.

Let f denote any member of S_0; that is, any one of the f_i. For each $k = 1, 2, \cdots, m$, there is some member of R whose σ distance from $f(x_k)$ does not exceed $\epsilon/4$; choose one such point and call it z_k. In this way we associate with f the finite sequence

$$z_1, z_2, \cdots, z_m .$$

Each z_k is one of the y's, and

$$\sigma(f(x_k), z_k) < \epsilon/4 .$$

Since there are only finitely many sequences of length m that can be formed from $y_1, y_2, \cdots, y_n$, some sequence must be repeated infinitely often; that is, all f in an infinite subsequence S of S_0 must be associated with the same $z_1, z_2, \cdots, z_m$. If f and g are two members of S and x is any point of D let x_i be a member of D such that $\rho(x, x_i) < S$. Then using the triangular inequality appropriately, one sees that

$$\sigma(f(x), g(x)) < \sigma(f(x), f(x_i)) + \sigma(f(x_i), z_i) + \sigma(z_i, g(x_i)) + \sigma(g(x_i), g(x)) < 4 \cdot \epsilon/4 = \epsilon .$$

The lemma is now proved. The remainder of the proof of the theorem is the same as in the earlier case.

If one wishes to assert that the subsequence S converges uniformly to something, then one should insist that R be complete, as well as precompact.

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SOME INTERESTING ALGEBRAIC IDENTITIES

William S. McCulley

1. Consider the identities

$$(1.1) \qquad (x_1^2+x_2^2)(y_1^2+y_2^2) = \begin{cases} (x_1y_1+x_2y_2)^2+(x_1y_2-x_2y_1)^2 \\ \\ (x_1y_1-x_2y_2)^2+(x_1y_2+x_2y_1)^2 , \end{cases}$$

of which the form (1.1a) may be found as the third exercise in the first chapter of Richard Bellman's new book, *Introduction to Matrix Analysis* [1]. It is instructive to trace the historical development of some of the applications and generalizations of these forms from their earliest appearance to modern times. We shall find these forms significant in that part of number theory called Diophantine analysis, as well as in relation to such algebraic structures as real and complex fields, division rings, vector spaces, quadratic forms, and analysis.

2. In addition to his famous theorem, Pythagoras (525 B.C.) [2] is credited with a formula for generating triples of integers satisfying the theorem. A second formula is attributed to Plato (375 B.C.). The first formal step in generalizing the formulas appears to have been taken by Euclid (300 B.C.) [2], though the Babylonians had prepared tables of Pythagorean triples more than a thousand years before Pythagoras [3]. Euclid, in his *Elements*, Book X, problem 28, lemma, proved, in substance, that if x_1 and x_2 denote relatively prime integers of opposite parity and

$$h = x_1^2+x_2^2 , \quad a = x_1^2-x_2^2 , \quad b = 2x_1x_2 ,$$

then

$$h^2 = a^2+b^2 .$$

This proof depends on the identity which we write in modern notation as

$$(2.1) \qquad (x_1^2+x_2^2)^2 = (x_1^2-x_2^2)^2+(2x_1x_2)^2 .$$

We see that (2.1) is a special case of (1.1b) obtained by setting $y_1 = x_1$, $y_2 = x_2$.

In Problem 9, Book II of his *Arithmetica* [4], Diophantos (1st Cent. A.D.) [3] obtained forms equivalent to (1.1) in determining conditions under which the hypotenuse of a Pythagorean right triangle can be the hypotenuse of a second distinct right triangle. Let us see how forms (1.1) solve this problem.

$$\begin{aligned} 65 &= 5\cdot 13 = (2^2+1^2)(3^2+2^2) = (2\cdot 3 \pm 1\cdot 2)^2+(2\cdot 2 \mp 1\cdot 3)^2 \\ &= 8^2+1^2 = 7^2+4^2 , \end{aligned}$$

and

$$65^2 = 63^2 + 16^2 = 33^2 + 56^2 .$$

Thus 65 is the hypotenuse of two Pythagorean triangles whose legs are 63 and 16, or 33 and 56, respectively. Pythagoras, Plato and Euclid were interested in positive integral solutions, whereas Diophantos sought positive rational solutions.

3. Identities (1.1) next appear eleven centuries later, in 1202 when Leonardo of Pisa, alias Fibonacci, published his book, *Liber Abaci*, [3] which contained a proof of the identities. About four centuries later, in 1621, Bachet published his translation of Diophantos's *Arithmetica*. Through this book Fermat came to know the problems that stimulated much of his work in number theory. Fermat used (1.1) [5] to prove that primes of the form $4n+1$ can be expressed as a sum of squares in exactly one way, and that the product of two primes, each of the form $4n+1$, is a sum of squares in two ways, as above, and so on. A further conjecture announced by Fermat, later proved by Euler, was "Every (positive) integer is a square, or is the sum of two, three or four squares." [4], [5].

About a century later, in 1748, Euler generalized the identities (1.1) to the forms [6]

$$(x_1^2 + x_2^2 + x_3^2 + x_4^2)(y_1^2 + y_2^2 + y_3^2 + y_4^2) = z_1^2 + z_2^2 + z_3^2 + z_4^2 , \tag{3.1}$$

where

$$z_1 = x_1y_1 + x_2y_2 + x_4y_4 , \quad z_2 = x_1y_2 - x_2y_1 \pm x_3y_4 \mp x_4y_3 ,$$

$$z_3 = x_1y_3 \mp x_2y_4 - x_3y_1 \pm x_4y_2 , \quad z_4 = x_1y_4 \pm x_2y_3 \mp x_3y_2 - x_4y_1 .$$

These forms led to Hamilton's development of quaternions, the elements of a four-dimensional vector space, in 1843. Two years later Cayley extended Hamilton's results, generalizing (1.1) to

$$\left(\sum_1^8 x_i^2\right)\left(\sum_1^8 y_i^2\right) = \sum_1^8 z_i^2 , \tag{3.2}$$

where the forms for z_i, similar to those of (3.1), are irrelevant to our purpose.

In 1898, A. Hurwitz put a stop to this line of generalization by proving the existence of such "reproducing forms" only for $n = 2, 4, 8$.

4. Let us review this line of development in terms of more recent mathematics, namely algebraic number fields. To fix ideas, consider the square of the modulus of a complex number [7]

$$|z_1|^2 = (x_1 + ix_2)(x_1 - ix_2) = x_1^2 + x_2^2 .$$

This is called the "norm" of z_1. The product z_1z_2 is

$$z_1z_2 = (x_1 + ix_2)(y_1 + iy_2) = (x_1y_1 - x_2y_2) + i(x_1y_2 + x_2y_1) ,$$

and the norm of the product is

$$|z_1 z_2|^2 = (x_1 y_1 - x_2 y_2)^2 + (x_1 y_2 + x_2 y_1)^2 ,$$

while the product of the norms is

$$|z_1|^2 |z_2|^2 = (x_1^2 + x_2^2)(y_1^2 + y_2^2) .$$

Thus, if we require that the "product of the norms be the norm of the product", we have (1.1b).

We recall that the complex numbers form a "field"; in particular, for the elements of a field, multiplication is both commutative and associative. For Hamilton's quaternions, however, multiplication is non-commutative, so quaternions form a "division ring" but not a field. [8] For the Cayley numbers multiplication is neither commutative nor associative.

To conclude this part of the discussion, we may note that forms like (1.1) or

$$\left(\sum_1^n x_i^2\right)\left(\sum_1^n y_i^2\right) = \sum_1^n z_i^2 , \quad n = 2, 4, 8,$$

are called "reproducing forms" [9], and that numbers expressed in these forms satisfy the condition that the norm of the product equals the product of the norms.

5. We observe that Diophantos and Fermat used both (1.1a) and (1.1b) for their number-theoretic purposes. Moreover, the form (1.1b) expresses an important property of complex and hypercomplex numbers. To trace the other, or what we would now call the vector, path of development, we go back to Lagrange [10], who, in 1773, proved the identity we would write as

$$(5.1) \qquad \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix}^2 = \begin{vmatrix} x_1^2 + x_2^2 & x_1 y_1 + x_2 y_2 \\ x_1 y_1 + x_2 y_2 & y_1^2 + y_2^2 \end{vmatrix} .$$

If we expand the left member of (5.1) using row-row multiplication, we obtain the right member. Then by expanding the determinant on the right and rearranging terms, we obtain the form (1.1a). What Lagrange proved was the corresponding identity involving the square of a third order determinant. In terms of Gibbs' vector notation, the Lagrange identity appears as [11], [12]

$$(5.2) \qquad (a \times b) \cdot (c \times d) = \begin{vmatrix} a \cdot c & a \cdot d \\ b \cdot c & b \cdot d \end{vmatrix} .$$

If we set $X = (x_1 x_2) = a = c$, $Y = (y_1 y_2) = b = d$ in (5.2), we get

$$(5.3) \qquad (X \times Y) \cdot (X \times Y) = \begin{vmatrix} X \cdot X & X \cdot Y \\ X \cdot Y & Y \cdot Y \end{vmatrix} .$$

Thus this special case of the Lagrange identity coincides with (1.1a). The first term in the right member of (1.1a) is recognizable as the square of the

inner product of the vectors $(x_1 x_2)$, $(y_1 y_2)$, while the second term is the square of the area of the parallelogram of which two adjacent sides are the same two vectors.

In 1821, Cauchy proved a similar identity [10]

$$(5.4)\qquad \left(\sum_1^n x_i^2\right)\left(\sum_1^n y_i^2\right) = \left(\sum_1^n x_i y_i\right)^2 + \sum_1^n (x_i y_j - x_j y_i)^2, \quad i < j,$$

which, for $n = 2$, is the same as (1.1a). We see that omitting the second sum on the right in (5.4) may diminish the right member, hence we have the Cauchy-Schwarz inequality

$$(5.5)\qquad \sum_1^n x_i^2 \ \sum_1^n y_i^2 \geqq \left(\sum_1^n x_i y_i\right)^2$$

important in the study of vector spaces and in analysis.

6. In summary, we see that these identities (1.1) may be said to have originated in geometrical algebra, passed through Diophantine analysis and quadratic forms, contributed to the development of complex fields and division rings, to vector spaces, analysis, and partial differential equations. Some conception of the amount of work engendered by these forms may be gained by scanning the second volume of Dickson's *History of the Theory of Numbers* [13].

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TEACHING OF MATHEMATICS

Edited by

Joseph Seidlin and C. N. Shuster

This department is devoted to the teaching of mathematics. Thus articles on methodology, exposition, curriculum, tests and measurements, and any other topic related to teaching, are invited. Papers on any subject in which you, *as a teacher*, are interested, or questions which you would like others to discuss, should be sent to *Joseph Seidlin, Alfred University, Alfred, New York.*

MULTIPLE INTERPRETATIONS OF SOME INTEGRALS

Verner E. Hoggatt

INTRODUCTION

In teaching the calculus sequence one talks about centroids, moments of inertia and radii of gyration; volumes of revolution; pressure and total force on submerged planar areas; moments of force and the center of fluid pressure and perhaps even the products of inertia. These integrals are related to one another. Collected here together are several isolated results which do not appear, to my knowledge, in any one book and a new result about the center of fluid pressure coordinate other than the depth of the center of fluid pressure, which is the one usually discussed. It is assumed that the necessary definitions of centroids; moments of area, volume, force and inertia; radius of gyration; and the center of fluid pressure are the usual ones.[3]

Suppose A is the area of one side of the enclosed region, R, lying wholly within the first quadrant, which is completely submerged in a fluid of uniform volume density, W. If the y-axis lies in the surface of the liquid and the positive x-axis is in the direction of gravity, then the fluid pressure per unit area at a depth x below the surface is given by Wx.[1]

The following integrals are meaningful. (See Figure 1.)

A. $$I_1 \int_{x_1}^{x_2} [y_2(x) - y_1(x)]\, dx$$

B. $$I_2 \int_{x_1}^{x_2} x[y_2(x) - y_1(x)]\, dx$$

C. $$I_3 \int_{x_1}^{x_2} \pi[y_2^2(x) - y_1^2(x)]\, dx$$

D. $$I_4 \int_{x_1}^{x_2} \pi x[y_2^2(x) - y_1^2(x)]\, dx$$

E. $$I_5 \int_{x_1}^{x_2} x^2[y_2(x) - y_1(x)]\,dx$$

NOTE: In all the aforementioned integrals a horizontal element is used.

FREE SURFACE OF LIQUID

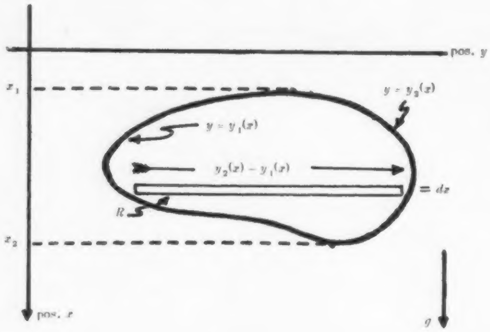

Figure 1.

We shall interpret some of these integrals in several ways:

1. I_1 is the area, A, of one side of submerged region, R.

2.(a) I_2 is the moment of area about the y-axis and, if $(\bar{x}, \bar{y})$ are the centroidal coordinates of area, A, then $I_2 = \bar{x}A$.

(b) If F is the total force on area, A, due to being submerged, then

(R$_1$) $$F = WI_2 = W\bar{x}A\ .$$

(c) If V_y is the volume of revolution about the y-axis (by the method of shells) of the area, A, then

(R$_2$) $$V_y = I_3 = 2\pi\bar{x}A\ .$$

3.(a) If V_x is the volume of revolution of area, A, about the x-axis (by the method of washers), then

(R$_3$) $$V_x = I_3 = 2\pi\bar{y}A\ .$$

(b) Therefore,

(R$_4$) $$\bar{y}V_y = \bar{x}V_x\ ,$$

from 2 (c) and 3 (a).

4.(a) If M_x is the moment of volume of volume, V_y, in 2 (c), (by the method of shells), and if $(O, \bar{\bar{y}})$ are tne volume centroidal coordinates of V_y, then

(R$_5$) $$M_x = I_4 = \bar{\bar{y}}V_y = \bar{\bar{y}}(2\pi\bar{x}A)\ ,$$

from 2 (c).

(b) If M_y is the moment of volume of volume V_x, in 3 (a), (by the method of washers), and, if $(\bar{\bar{x}}, O)$ are the volume centroidal coordinates of V_x, then

(R$_6$) $$M_y = I_4 = \bar{\bar{x}} V_x = \bar{\bar{y}} 2\pi \bar{y} A = \bar{\bar{y}} 2\pi \bar{x} A = \bar{\bar{y}} V_y \ ,$$

from 3 (a).

NOTE: From R$_6$: $\bar{\bar{x}}\bar{y} = \bar{\bar{y}}\bar{x}$; then if area, A, has symmetry about $y = \bar{y}$, so that $\bar{y} = \bar{\bar{y}}$, then $\bar{x} = \bar{\bar{x}}$ without symmetry.

(c) If T_x is the moment of force about the x-axis of one side of submerged region, R, then

(R$_7$) $$T_x = \frac{WI_4}{2\pi} = \frac{WM_x}{2\pi} = \frac{W\bar{\bar{y}}(2\pi\bar{x}A)}{2\pi} \ ,$$

from 3 (b) and 4 (a).

If $(\bar{\bar{\bar{x}}}, \bar{\bar{\bar{y}}})$ are the coordinates of the center of fluid pressure, then $T_x = \bar{\bar{\bar{y}}}F$, where F is the total force on one side of submerged region, R. But, from 2 (b), $F = W\bar{x}A$ and $\bar{\bar{\bar{y}}} = T_x/F$, hence

(R$_8$) $$\bar{\bar{\bar{y}}} = \frac{W\bar{\bar{y}}(2\pi\bar{x}A)}{2\pi F} = \frac{\bar{\bar{y}}(2\pi W\bar{x}A)}{2\pi(W\bar{x}A)} = \bar{\bar{y}} \ .$$

(d) $(1/2\pi)I_4$ expressed as a double integral is called the product of inertia.[2]

(e) The work to pump the liquid in volume, V_x, up to the liquid surface is $W(ork) = WI_4 = \bar{x}(WV_x)$, a result well known to physicists.

5.(a) If I_y is the moment of inertia, about the y-axis, of area, A, then $I_y = k^2 A = I_5$, where k is the radius of gyration of area, A, about the y-axis.

(b) If T_y is the moment of force about the y-axis due to pressure on the submerged area, A, then $T_y = \bar{\bar{\bar{x}}}F$, where F is the total force on one side of the submerged area, A. Therefore $T_y = WI_5 = Wk^2A$ and

(R$_9$) $$\bar{\bar{\bar{x}}} = \frac{T_y}{F} = \frac{Wk^2A}{W\bar{x}A} = \frac{k^2}{\bar{x}} \ ,$$

from 2 (b).

SUMMARY

Result R$_1$ is a useful result for fluid mechanics and is well known.

Results R$_2$ and R$_3$ show a theorem of Pappus for revolving a plane area about an axis not cutting the plane area.

Results R$_4$ and R$_9$ are usually interesting problems in some books.

Result R$_5$ follows from:

$$M_x = \int_{x_1}^{x_2} \underbrace{\tfrac{1}{2}(y_2(x) + y_1(x))}_{\substack{\text{Lever arm to} \\ \text{centroid of the} \\ \text{volume shell}}} (\underbrace{2\pi x[y_2(x) - y_1(x)]}_{\text{Volume shell}})\, dx\,.$$

Result R_6 follows from:

$$M_y = \int_{x_1}^{x_2} \underset{\text{Lever}}{x} \underbrace{(\pi)(y_2^2(x) - y_1^2(x))}_{\text{Volume washer}}\, dx\,.$$

Result R_7 is believed to be new: If $\bar{\bar{\bar{y}}}$ is the y-coordinate of the center of fluid pressure and $\bar{\bar{y}}$ is the y-coordinate of the volume centroid of V_y, obtained by revolving area, A, about the y-axis, then $\bar{\bar{\bar{y}}} = \bar{\bar{y}}$.

Result R_8 follows from:

$$T_x = \int_{x_1}^{x_2} \underbrace{\tfrac{1}{2}(y_2(x) + y_1(x))}_{\substack{\text{Lever arm to} \\ \text{center of} \\ \text{pressure of} \\ \text{horizontal} \\ \text{element}}} \underbrace{(\underbrace{Wx}_{\substack{\text{Pres-} \\ \text{sure at} \\ \text{depth } x}} \underbrace{[y_2(x) - y_1(x)]\, dx}_{\substack{\text{Area of} \\ \text{horizontal} \\ \text{elements}}})}_{\text{Force on element}}\,.$$

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A PROBLEM ABOUT GEODESICS

A. R. Amir-Moéz

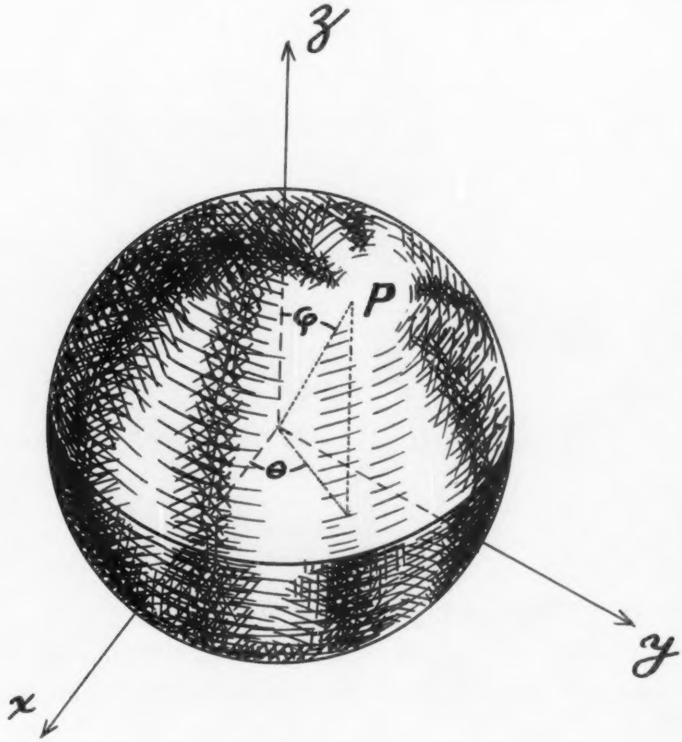
Geodesics of a sphere are its great circles. This is quite obvious. As a matter of fact, in many books, the previous sentence is the proof of the proposition given in the first sentence. Sometimes a differential equation for the geodesics of a sphere is given without formal solution. In some books a few special great circles of the sphere are tried in the equation and then it is said, "Well! It isn't worth solving the equation."

It is true. Proving an obvious fact as this is a waste of time. But techniques used in this note may be of some interest. Suppose we choose the sphere

$$x^2+y^2+z^2=r^2 .$$

A parametric form of this equation is:

$$\begin{cases} x = r\sin\phi\cos\theta \\ y = r\sin\phi\sin\theta \\ z = r\cos\phi . \end{cases}$$



Now

$$\begin{cases} dx = r[\cos\phi\cos\theta\, d\phi - \sin\phi\sin\theta\, d\theta] , \\ dy = r[\cos\phi\sin\theta\, d\phi + \sin\phi\cos\theta\, d\theta] , \\ dz = -r\sin\phi\, d\phi . \end{cases}$$

Thus

$$ds^2 = r^2[(d\phi)^2 + \sin^2\phi\,(d\theta)^2] .$$

The length of a curve on the sphere between two points has the form

$$s = r\int_{\theta_1}^{\theta_2} \sqrt{\phi'^2 + \sin^2\phi}\, d\theta ,$$

where ϕ is taken a function of θ, and $\phi' = d\phi/d\theta$.

In order to minimize this we use Euler equation [1]. Let

$$f(\theta) = \sqrt{\phi'^2 + \sin^2\phi} .$$

The Euler equation is

$$\frac{d}{d\theta}\Big[\frac{\partial f}{\partial \phi'}\Big] = \frac{\partial f}{\partial \phi} .$$

$$\frac{\partial f}{\partial \phi'} = \frac{\phi'}{\sqrt{\phi'^2 + \sin^2\phi}} ,$$

and

$$\frac{d}{d\theta}\Big(\frac{\partial f}{\partial \phi'}\Big) = \frac{\phi''\sqrt{\phi'^2 + \sin^2\phi} - \dfrac{\phi'(\phi'\phi'' + \phi'\sin\phi\cos\phi)}{\sqrt{\phi'^2 + \sin^2\phi}}}{\phi'^2 + \sin^2\phi}$$

$$= \frac{\phi''\phi'^2 + \phi''\sin^2\phi - \phi''\phi'^2 - \phi'^2\sin\phi\cos\phi}{(\phi'^2 + \sin^2\phi)\sqrt{\phi'^2 + \sin^2\phi}} .$$

Then

$$\frac{\partial f}{\partial \phi} = \frac{\sin\phi\cos\phi}{\sqrt{\phi'^2 + \sin^2\phi}} .$$

Thus the differential equation of geodesics is :

$$\phi''\sin^2\phi - \phi'^2\sin\phi\cos\phi = \phi'^2\sin\phi\cos\phi + \sin^3\phi\cos\phi .$$

We see that $\sin\phi = 0$ does not give anything interesting. Let $\sin\phi \neq 0$. Then we have

$$\phi''\sin\phi - 2\,\phi'^2\cos\phi - \sin^2\phi\cos\phi = 0 . \tag{1}$$

Ordinarily this equation is solved as follows : Since the independent variable is not present we choose $\phi' = t$ and change the equation to a first

order differential equation, etc. But we shall use the following technique. Let $\operatorname{ctn}\phi = u$. Then

(2) $$-\phi' \csc^2\phi = u'$$

all derivatives are with respect to θ. We take the second derivative; we get

(3) $$-\phi'' \csc^2\phi - 2\phi'^2 \csc^2\phi \operatorname{ctn}\phi = u''.$$

We solve (2) and (3) for ϕ' and ϕ''.

$$\phi' = -u' \sin^2\phi .$$

Then we have

$$-\phi'' \csc^2\phi = u'' - 2u'^2 \sin\phi\cos\phi ,$$

or

$$\phi'' = 2u'^2 \sin^3\phi\cos\phi - u'' \sin^2\phi .$$

Substituting ϕ' and ϕ'', in (1), in terms of u we get

$$2u'^2\sin^4\phi\cos\phi - u''\sin^3\phi - 2u'^2\sin^4\phi\cos\phi - \sin^2\phi\cos\phi = 0 .$$

Since $\sin\phi \neq 0$ we have

$$u''\sin\phi + \cos\phi = 0 , \quad \text{or} \quad u'' + u = 0 .$$

The solution of this linear differential equation is

$$u = a\cos\theta + b\sin\theta .$$

Thus we have

$$\operatorname{ctn}\phi = a\cos\theta + b\sin\theta , \quad \text{or} \quad \cos\theta = a\cos\theta\sin\phi + b\sin\theta\sin\phi .$$

Multiplying through by r we get

$$r\cos\phi = ar\cos\theta\sin\phi + br\sin\theta\sin\phi ,$$

or

$$z = ax + by .$$

That is a geodesic of the form

$$\begin{cases} x^2 + y^2 + z^2 = r^2 \\ z = ax + by , \end{cases}$$

which is a great circle.

---

[1] *The Tree of Mathematics.* Glenn James, The Digest Press, 257 Tally-Ho Rd., Arroyo Grande, California (1957), pp. 284-304.

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PALINDROMIC CUBES

C. W. Trigg

On page 45 of J. Newton Friend's *Numbers, Fun & Facts*, Charles Scribner's Sons, (1954) the following statement appears :

"Primes are the only numbers which, when cubed, yield palindromes, that is, numbers reading the same backwards as forwards. No composite number can do so. E. g., $11^3 = 1331$."

It is true that

$$7^3 = 343$$

and

$$101^3 = 1030301 \, .$$

However,

$$(111)^3 = (3 \cdot 37)^3 = 1367631 \, ,$$

and any number of the form

$$N = 10^k + 1$$

gives a palindromic cube consisting of $k-1$ zeros between each consecutive pair of 1, 3, 3, 1. Now if $k = 2n+1$, $n > 0$, then N is divisible by 11 and hence composite. E.g.,

$$(1001)^3 = (7 \cdot 11 \cdot 13)^3 = 1003003001 \, .$$

Furthermore,

$$10001 = 73 \cdot 137$$

and

$$1000001 = 101 \cdot 9901 \, .$$

The cubes of these composite numbers are also palindromes.

On page 103 of the same book it is stated that "Eleven is unique in that its second, third and fourth powers are palindromes."

Now N^2 and N^4 are also palindromes. E.g.,

$$(1001)^2 = 1002001$$

and

$$(1001)^4 = 1004006004001 \, .$$

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TRANSFORMATIONS AND OPTICS

George M. Bergman

The principals of elementary lense optics, as taught in high school physics, may be formalized as follows:

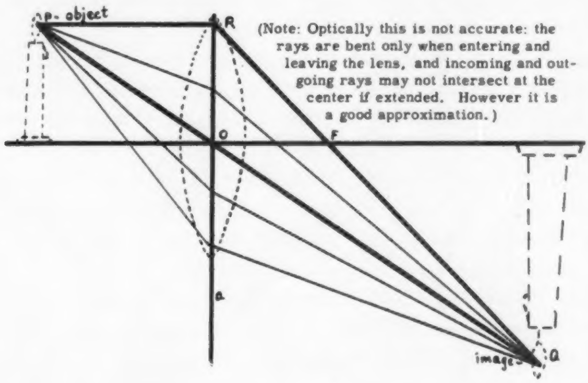
I. An ideal lense a is a line (we deal with a plane diagram, a cross-section) to which corresponds a one-to-one transformation of the class of all lines in the plane into itself, such that every line cutting a is concurrent with its transform and a.

II. There is a point O on a such that the transform of any line through O is itself. (i.e., rays of light through the "optical center" aren't bent.)

III. We are given another point F such that all lines parallel to OF are transformed into lines through F (i.e., a beam of light parallel to the "principal axis" is focussed upon the "principal focus".)

IV. The transform of the pencil of lines through any point P, the "object" (real or "at infinity") is the pencil through a point Q, its "image".

(To construct Q given P not on OF, the student finds the transform of $PR \mid\mid OF$, and of PO, and constructs their intersection; other lines of the two pencils may then be drawn – see diagram.)



I, II, and III are obviously consistent, but are insufficient to define the transformation. Does the addition of IV yield a complete and consistent determination?

In order to answer this question let us project the plane of our diagram onto another, so that FO becomes the line at infinity. Axioms II and III are now:

II.′ All lines in the pencil y of lines parallel to the lens are transformed into themselves.

III.′ A certain pencil of parallels x (lines through what was OF's point at

infinity) is transformed into another pencil of parallels x' (through F).

The location of a point P in the unprojected diagram in terms of the line through it parallel to OF and the line PO, and of an image Q in terms of QF and QO, are now locations of P and Q in the (oblique or rectangular) coordinate systems (x, y), and (x', y). Though their axes are not scaled, we know that corresponding lines of x and x' must be proportionately spaced, as they meet on the lens (by I); and the ordinates are identical by II. Therefore, a locus giving a line in one coordinate system will give a line in the

The "student's construction" locates the image of a point as having the same coordinates in (x', y) as the point had in (x, y). The transform of a line is the line drawn through transforms of two of its points, and will have in the new system the original equation since it passes through lines with the same coordinates. The images of points at infinity (points on OF, not determined by our construction) can be found as the points at infinity of the transforms of lines through the originals.

Thus the transformation is completely determined from the axioms, preserves incidence, and, it is easy to see, contradicts none of the axioms. As the hypotheses, constructions, and conclusions of our argument involve only points, lines, and incidence, they are applicable to the original plane, and the question is answered in the affirmative.

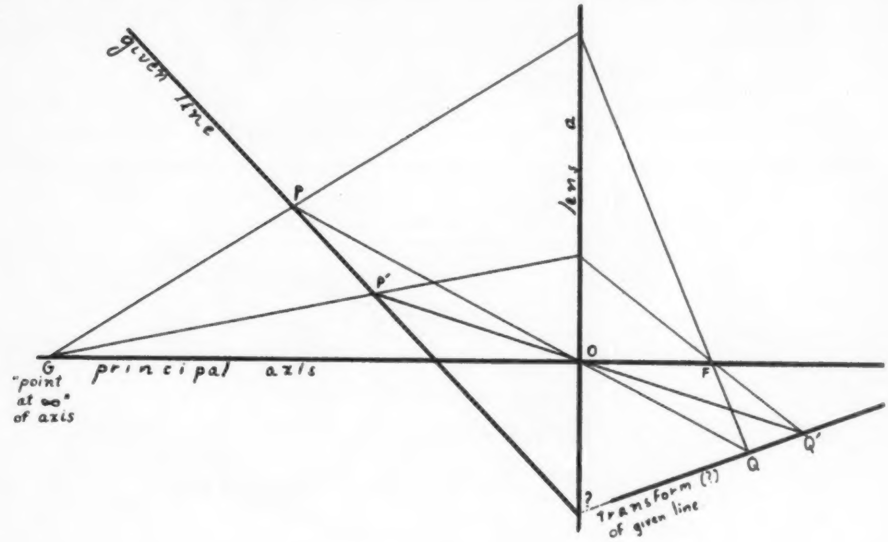
An alternate way to approach the question would be to proceed as follows. As all lines not parallel to a are the transforms of lines not parallel to a, and as the transformation is one-to-one, the transforms of all lines parallel to a must be lines parallel to a. Thus I implies that it holds even for lines that "cut" a at infinity. As neither II, III, or IV treats exceptionally the line at infinity, we may consider ourselves dealing with the projective plane. Now III merely says that we are given the image F of a point, which we may call G, on OF.

It will now suffice to find whether from our axioms we can derive a one-to-one correspondence of lines with lines *and points with points* (object with image) preserving incidence and obeying I, II, and III.

We have already defined a unique point-to-point correspondence (object-image) for points *not on OF*. Note that it is one-to-one and onto, for it has an inverse: we can carry through the construction in reverse. If we can now find a transformation of all lines, consistent with it, we can use this to extend it onto OF. For, given any point H on OF, the transforms of all lines through H must all meet at some point K on OF – for none of them meet anywhere off OF. K will then be our one and only possible choice for the transform of H. By the same reason as before, this extended transformation will be one-to-one and onto. So we must see whether we can derive from the axioms a line-to-line transformation, consistent with them, and so also with the point-image construction.

Now, a line's transform should pass through its intersection with a and through the image Q of one of its points P; this gives us a line-transform construction. The consistency of this method, and so of our whole

system, depends on whether we get the same result using another point, P' on the line; which is equivalent to asking whether the line through the images, QQ' is concurrent with a and PP'. But drawing the diagram, we see that it represents Desargue's theorem, a basic principal of projective geometry! Joins of corresponding vertices of triangles FPP' and GQQ' concur



(at O) so the intersections of corresponding sides will be colinear (on a, for that's where FP meets GQ, and FP' GQ'.)

Which, as we've shown, makes the answer to the question, "Yes."

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EVALUATION OF DETERMINANTS BY CONGRUENCIES

Ernest W. Valyocsik

We desire to prove graphically that the parallelogram formed by the vectors $V_1 = a_{11}i + a_{12}j$ and $V_2 = a_{21}i + a_{22}j$ has an area equal to the area of the parallelogram formed by the vectors $W_1 = a_{11}i + a_{21}j$ and $W_2 = a_{12}i + a_{22}j$. We will also prove graphically that each of these areas is equal to the determinant

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12} .$$

Let us suppose we form the parallelogram in Figure 1 with the vectors

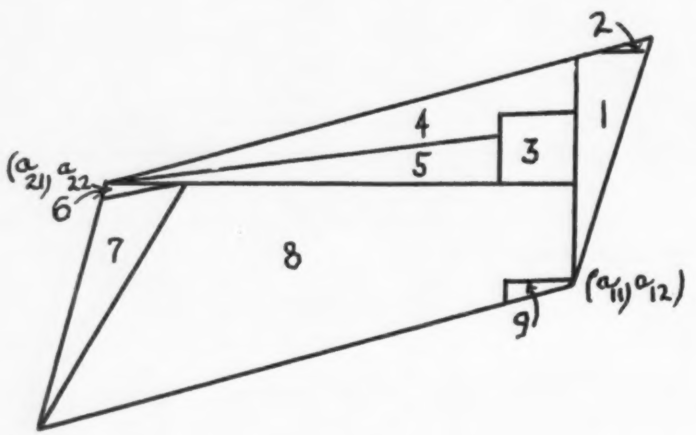

Figure 1

V_1 and V_2. The numbered sections enclosed by the heavy lines show the cuts that are necessary to form the new rectangles and parallelogram from the original parallelogram. There are nine sections in all.

The intersections of the perpendiculars dropped from the points (a_{11}, a_{12}) and (a_{21}, a_{22}) produce a large and small rectangle. The small rectangle is taken from the left corner to minimize the number of cuts necessary in the original parallelogram. The numbered sections are shown in Figure 2 in their new arrangement. This demonstrates that the area of the original parallelogram of Figure 1 is equal to the area of the large rectangle minus the area of the small rectangle. Expressed algebraically, the area of the

original parallelogram equals

$$(a_{22}-a_{12})a_{11}+(a_{11}-a_{21})a_{12} = a_{22}a_{11}-a_{12}a_{11}+a_{11}a_{12}-a_{21}a_{12}$$
$$= a_{22}a_{11}-a_{21}a_{12}\ .$$

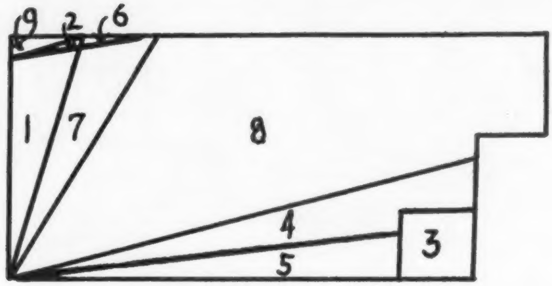

Figure 2

By moving section 3 to the new position shown in Figure 3, we produce a region with area

$$(a_{22}-a_{21})a_{11}+(a_{11}-a_{12})a_{21} = a_{22}a_{11}-a_{21}a_{11}+a_{11}a_{21}-a_{12}a_{21}$$
$$= a_{22}a_{11}-a_{21}a_{12}\ .$$

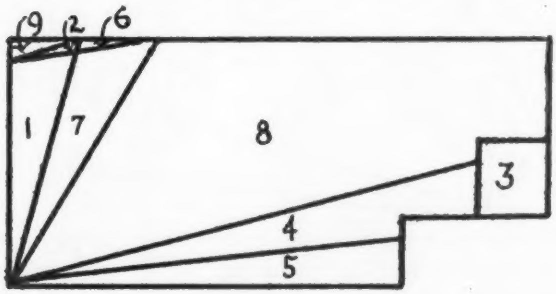

Figure 3

We see that the areas pictured in Figures 2 and 3 are equal.

One is now able to generate the parallelogram, Figure 4, whose area is equal to the area of the region in Figure 3. This parallelogram is formed by the vectors

$$W_1 = a_{11}i+a_{21}j \quad \text{and} \quad W_2 = a_{12}i+a_{22}j\ .$$

Since the area of the parallelogram in Figure 1 is equal to the area of the parallelogram in Figure 4, we may conclude that we have proven that the areas of the two parallelograms are equal.

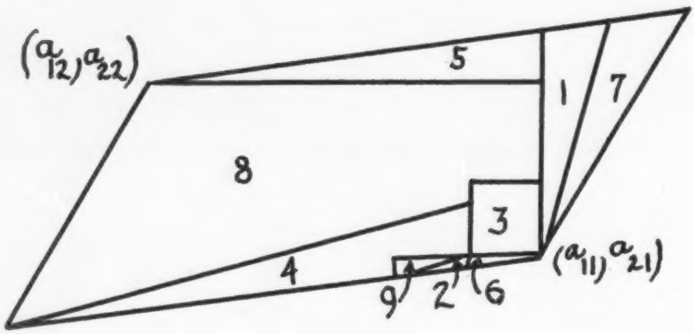

Figure 4

Figure 5 shows how a model can be constructed for demonstration purposes.

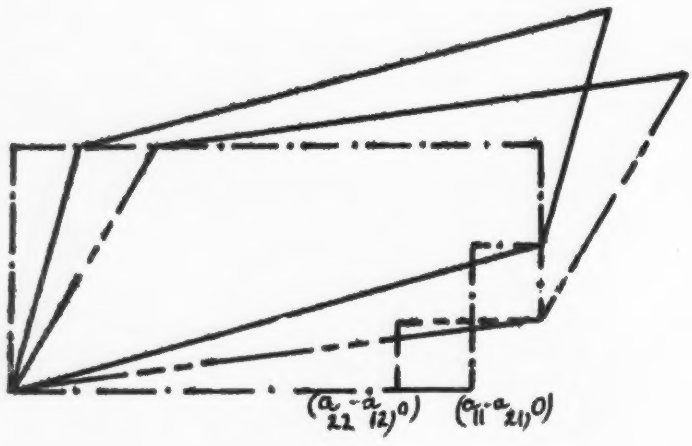

Figure 5

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A GRAPHICAL DETERMINATION OF THE NATURE OF THE ROOTS OF A CUBIC

Morton J. Hellman

Since any complete cubic can be reduced, it is sufficient to consider

(1) $$x^3 + ax + b = 0 \qquad (a,\ b \text{ real}) .$$

All the roots of (1) will be given graphically by the intersection of the curves

(2) $$y = x^3 \quad \text{and}$$

(3) $$y = -ax - b .$$

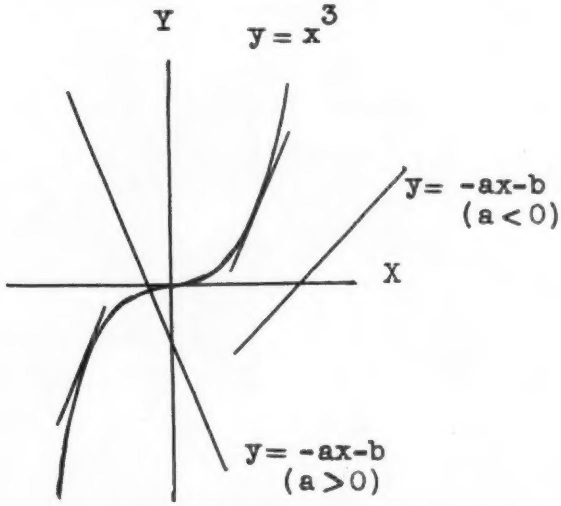

Figure 1

It is clear that any straight line parallel either to the X-axis or the Y-axis can intersect the curve (2) in only one point. Consider the remaining case where a is finite and non-zero shown in Fig. 1. If the straight line (3) has slope $-a$ ($a<0$), it is generally shown by the line at the right. If in (3) $a > 0$, the slope of the line is negative, and such a line can cut the graph of (2) in only one point. Hence, when $a > 0$, (1) has one real root and two imaginary roots, a result which follows immediately from Descartes' Rule of Signs. From Fig. 1 it follows that no straight line of positive slope can be drawn which fails to intersect (2) at least once. Hence, the usual result that every cubic equation must have at least one real root. Furthermore, Fig. 1 shows the impossibility of (3) intersecting (2) in only two real points (tangency points counted twice). Thus, (1) has either one or three real roots.

In Figure (1) consider a line $y = -ax - b$ ($a<0$) which is tangent to (2).

Since the slope of the tangent line to $y = x^3$ is $3x^2$ (which is always positive except at the origin), the two tangent lines of positive slope indicate the extreme positions that a line of that general slope, $-a$, $(a<0)$ must have if it is to cut (2) in three points.

Figure 2 shows the details of this case more completely. To determine the coordinates of the points of tangency, P and Q, set $3x^2 = -a$.

Thus, $x = \pm\sqrt{-a/3}$, and since P and Q are symmetric points with respect to the origin, the argument for point P will suffice. There, $(a < 0)$, $x = +\sqrt{-a/3}$, and equating the ordinates of (2) and (3) for this value of x, and simplifying

$$(b^2/4) + (a^3/27) = 0 \ . \tag{4}$$

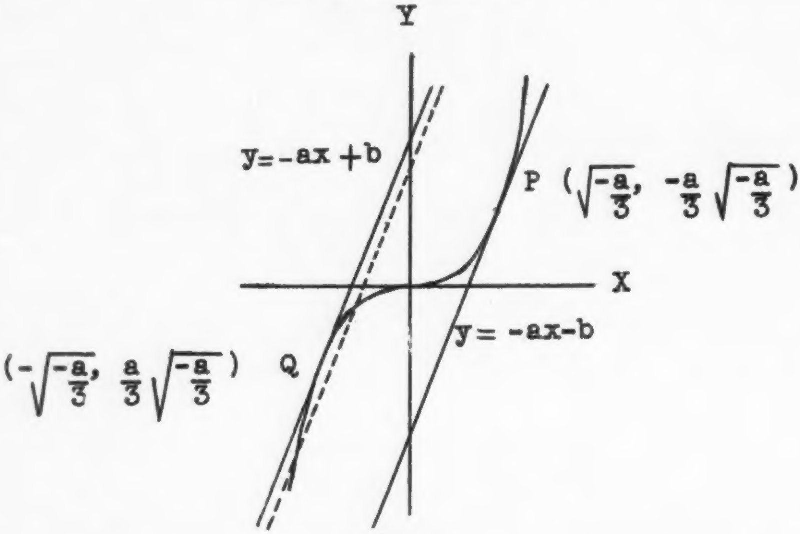

Figure 2.

In this tangency case, (2) and (3) must intersect at a point other than P and Q. This follows since the slope of (2) is $3x^2$ which assumes all real positive values as $|x|$ increases while the slope of the tangent lines is fixed.

The case of three real distinct roots corresponds to that of the dotted line parallel to the tangent lines at P and Q; in this case, the y-intercept is always less in absolute value than the y-intercept of the tangent lines, or

$$|-b| < |(-2a/3)\sqrt{-a/3}| \quad \text{which reduces to} \tag{5}$$

$$(b^2/4) + (a^3/27) < 0 \ . \tag{6}$$

The case of only one real root corresponds to the line lying outside of the limiting tangent lines, and this reverses the sense of the inequality (5). Hence, in this case,

$$(b^2/4) + (a^3/27) > 0 \ . \tag{7}$$

Summarizing these results, we have for $a < 0$, $D = (b^2/4) + (a^3/27)$, $D = 0$ two equal real roots, third root real, $D < 0$ three distinct real roots, $D > 0$ one real root, two imaginary roots.

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MISCELLANEOUS NOTES

Edited by

Charles K. Robbins

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SOME NOTES ON THE MEAN DIFFERENCE

John P. Gill

In this paper the mean difference formula is introduced and Tukey's concept of *inherited on the average* is used to show the mean of all possible sample mean differences is the population mean difference. There is also established a relationship between the standard deviation and the mean difference in a normal population. Experimental computations are given to show the relationship between these two measures for approximately normal finite populations.

In a recent paper [3] the author used the following notations in discussing the mean difference formula:

$$d_a^b = x_b - x_a, \quad x_b \geqq x_a, \tag{1}$$

$$d_1^N = x_N - x_1, \quad d_2^{N-1} = x_{N-1} - x_2,$$

where the variates are arranged in an array $x_1, x_2, x_3, \cdots, x_N$, and x_N is the largest in an algebraic sense.

$$D = \frac{D_N}{\frac{1}{2}N(N-1)} \tag{2}$$

is called the mean difference, where D_N represents the sum of all possible positive differences of pairs of N variates, and

$$D_N = (N-1)d_1^N + (N-3)d_2^{N-1} + \cdots + kd_{|\frac{1}{2}N|}^{|\frac{1}{2}N|+k}$$

where $k = 1$ when N is even, and $k = 2$ when N is odd; and $|\frac{1}{2}N|$ is the greatest integer contained in $\frac{1}{2}N$.

Theorem: The arithmetic average of the sample D's [designate a sample D based on S variates by D(s)] based on all possible combinations of a finite population of N variates chosen S at a time ($S \leqq N$) is equal to the mean difference, D, of the N variates of the population.

According to Tukey's argument [1], each d_i^j, $i \leqq j$, i, $j = 1, 2, \cdots, N$, will appear in the same number, $\binom{N-2}{S-2}$, of samples, and the population

mean difference, D, will be *inherited on the average* from the sample mean differences, $D(s)$.

Nair [2] proved, among other propositions, the following: For a normal parent

$$dF = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-x^2}{2\sigma^2}} dx\,, \quad -\infty \leqq x \leqq +\infty\,, \quad E[D(s)] = \frac{2\sigma}{\sqrt{\pi}}\,.$$

Therefore, by the above theorem, for a finite near-normal population the mean difference

$$D \doteq \frac{2\sigma}{\sqrt{\pi}}\,. \tag{3}$$

The writer computed D from a table of normal frequencies starting with $T = -4.2$ and proceeding by intervals of 0.1 to $T = +4.2$, and used a total frequency of $N = 100{,}000$. The result was $D = 1.128313$. Since in this distribution, $\sigma = 1$, formula (3) above becomes $D = (2\times 1)/\sqrt{\pi} = 1.128377$, with a relative error of 0.006%.

For less normal distributions the following table was used [4]:

								Total	M	σ	α_3	Actual	Est.	Error
X	60	65	70	75	80	85	90					D	D	
F	0	12	24	28	24	12	0	100	75	6	0	6.79	6.77	0.3%
F	4	4	20	44	20	4	4	100	75	6	0	6.34	6.77	7.0%
F	4	8	20	24	40	4	0	100	75	6	1.2	6.53	6.77	3.7%

An example of the computation of D follows:

x	f					
60	4	4				
65	8		8			
70	20			20		
75	24				12	12
80	40		8	20	12	
85	4	4				
90	0					
Total	100					

Using formula (2)

$$D = \frac{(99+97+\cdots+93)(25)+(91+89+\cdots+77)(15)+(75+73+\cdots+37)(10)}{\tfrac{1}{2}(100)(99)}$$

$$+\frac{(35+33+\cdots+15)(5)+0}{\tfrac{1}{2}(100)(99)}$$

Using the summation formula for an arithmetic progression with difference equal to -2, the initial multipliers in each of the products of the numerator above can be summed quickly, and

$$D = \frac{\frac{4}{2}(99+93)(25) + \frac{8}{2}(91+77)(15) + \frac{20}{2}(75+37)(10) + \frac{12}{2}(35+13)(5) + 0}{\frac{1}{2}(100)(99)}$$

$$= \frac{32320}{4950} = 6.53 .$$

Suggestions for possible further study could be to learn more about the degree of error in estimating D from σ in approximately normal distributions with various degrees of kurtosis and possibly the range of a distribution.

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2. Kendall, M. G., *The Advanced Theory of Statistics*, Vol. 1, London, Charles Griffin and Company, 1943, pp. 42-48.
3. Gill, John P., "Elementary Concepts of Functional Means and Dispersions," *Mathematics Magazine*, Vol. 24, (1950), pp. 65-75.
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University of Alabama

SENSIBLE ALTITUDES

Oh, come gather for festivals! Midgets, arise:
It's recorded the low shall be tall!
Yes, the infinitesimals smallest in size
Have their orders the highest of all.

A NOTE ON THE INDICAL EQUATION

William Squire

INTRODUCTION

A useful technique for solving linear differential equations is the Frobenius series expansion in which it is assumed that the form of the solution is

$$y = (x-a)^c \sum_{k=0}^{k=\infty} d_k(x-a)^k \ . \tag{1}$$

The coefficients are determined by substituting (1) into the differential equation and settingthe coefficients of each power of $(x-a)$ equal to zero. The exponent c of the prefactor is found from what is called the indical equation which is usually obtained by considering the lowest power of $(x-a)$. The object of this note is to present a method for finding c based on L'Hospital's rule [1] for evaluating an indeterminate expression.

While the general nth order linear differential equation

$$y^n = \sum_{k=0}^{k=n-1} p_k(x)y^k$$

has n linearly independent solutions, there may be less than n distinct values of c. There may be solutions which cannot be represented as power series or there may be several solutions which have the same value of c. Most tests treat the case of multiple roots by the familiar method of using a known particular solution to lower the order of a differential equation. There is however, a more elegant method based on the fact [2] that for linear equations if $y(x, c)$ is the solution of the form assumed in (1) and c is a root of multiplicity k; then additional independent solutions are

$$\frac{\partial y}{\partial c}, \quad \frac{\partial^2 y}{\partial c^2}, \quad \cdots, \quad \frac{\partial^{k-1} y}{\partial c^{k-1}} \ .$$

The corresponding leading terms are

$$(x-a)^c(\ln x{-}a)\ , \quad (x-a)^c(\ln x{-}a)^2\ , \quad \cdots, \quad (x-a)^c x(\ln x{-}a)^{k-1}$$

so that a slight generalization of the form assumed in (1) is involved.

DESCRIPTION OF THE METHOD

The basic idea of the present method is that

$$c = \lim_{x\to a} \frac{(x-a)y'}{y}$$

except for the case of a single root for which $c = 0$. By application of L'Hospital's rule, i.e. differentiation of the numerator and denominator

$$c = 1 + \lim_{x \to a} \frac{(x-a)y''}{y'} .$$

Further applications give

$$c = 1 + \lim_{x \to a} \frac{(x-a)y^n}{y^{n-1}} .$$

The differential equation expresses y^n in terms of the lower derivatives. The value of terms such as y^k/y^{n-1}, where $k < n-1$, is related to c by using

$$\frac{y^k}{y^{n-1}} = (x-a)^{n-k-2} \prod_{j=k}^{j=n-2} \frac{y^j}{(x-a)y^{j+1}} .$$

The degree of the resulting algebraic equation cannot exceed the order of the differential equation and may be less if

$$\lim_{x \to a} (x-a)^{n-2} p_0(x) = 0 .$$

ILLUSTRATIVE PROBLEMS

The procedure is best understood by working some examples. Without loss of generality, we can consider only expansions around $x = 0$ as the point around which the solution is expanded can be made the origin by a shift when a is finite. When the expansion is around the point at infinity, a new variable such as $1/x$ accomplishes the change.

For the familiar equation

$$y'' = ky$$

the exponent is found by

$$c = 1 + \lim_{x \to 0} \frac{xy''}{y'} = 1 + \lim_{x \to 0} \frac{kx^2}{c} = 1 .$$

This exponent corresponds to the sinh (or sin) solution. The zero exponent corresponding to the cosh (or cos) solution is lost as it is a single root.

For Bessel's equation

$$y'' = \left(\frac{n^2}{x^2} - 1\right)y - \frac{y'}{x}$$

the indical equation is

$$c = 1 + \lim_{x \to 0} \frac{x}{y}\left[\left(\frac{n^2}{x^2} - 1\right)y - \frac{y'}{x}\right] = \lim_{x \to 0} \frac{n^2 - x^2}{c} .$$

There are two solutions, $c = +n$ and $c = -n$, except when $n = 0$. In that

case, $c = 0$ and the leading terms of the two independent solutions are x^0 and $\ln x$.

The equation

$$y'' = \frac{2y}{x^3} - \frac{1-2x}{x^2}y'$$

has one solution whose leading term is x^2 and a second solution with no series expansion around the origin [3]. Application of the present method gives

$$c = 1 + \lim_{x\to 0} \frac{2-c}{x}.$$

Clearly, this is satisfied if c is infinite, corresponding to the solution which has no series expansion. By transforming the equation into

$$c = \frac{\lim\limits_{x\to 0} 3 + \frac{2}{x}}{\lim\limits_{x\to 0} 1 + \frac{1}{x}}$$

the $c = 2$ solution is obtained.

A nonlinear example.

The application of the method to nonlinear equations is best discussed by working an example. The equation

(2) $$y''' = -\tfrac{1}{2} yy''$$

occurs in the theory of the boundary layer on a flat plate [4]. As it is a third order equation, the general solution involves three arbitrary constants. However, the solution is not a linear function of the three constants. Furthermore when there are multiple roots of the indical equation $\partial y/\partial c$ is not necessarily a second solution.

While the general solution of (2) is not known, four particular solutions involving a single constant can be written down. They are:

(3) $$y = b_1$$

for which $c = 0$;

(4) $$y = b_2 x,$$

for which $c = 1$;

(5) $$y = \frac{b_3 x^2}{2!} - \frac{b_3^2 x^5}{2\times 5!} + \frac{11 b_3^3 x^8}{4\times 8!} - \cdots$$

for which $c = 2$; and

(6) $$y = \frac{6}{b_4 + x}*,$$

for which $c = 0$ if $b_4 \neq 0$ and $c = -1$ if $b_4 = 0$.

[*To the author's knowledge (6) has not been reported in the literature. This may be because it has no simple physical interpretation as a flow.]

While the sum of (3) and (4) is a solution, no other additive combination is.

Application of our technique gives

$$c = 2 - \frac{1}{2} \lim_{x \to 0} xy \ .$$

It is characteristic of nonlinear equations that the indical equation involves the value of the function or its derivatives at the point around which it is being expanded. Now if

$$\lim_{x \to 0} xy = 0 \ ,$$

then $c = 2$. This corresponds to (4). On the other hand if the limit is finite, which will only happen for $c = -1$,

$$\lim_{x \to 0} xy = 6 \ .$$

This corresponds to (6) with $b_4 = 0$. The loss of one $c = 0$ root is to be expected but the loss of the second one and the loss of the $c = 1$ root corresponding to (4) indicates that the method will not give all the values of c for a nonlinear equation.

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2. H. T. H. Piaggio, *An Elementary Treatise on Differential Equations*, London, 1940, p. 113.

3. Ibid, p. 118.

4. H. Schlichting, *Boundary Layer Theory*, New York, 1955, p. 104.

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LINE INTEGRALS IN A MULTIPLY CONNECTED REGION

Arthur B. Brown

The purpose of this note is to prove, with no demands on topology other than the property that the interior of a sphere is simply connected, the known theorem that conditions (2) below imply the invariance of the line integral (1) in a multiply connected region under deformations leaving the end points fixed, assuming the standard theorem for simply connected regions. A similar result holds for two closed curves, one of which can be deformed into the other.

Let $P_1(x_1, \cdots, x_n), \cdots, P_n(x_1, \cdots, x_n)$, $n \geqq 2$, be real single-valued functions of class C^1 in an open set R of $x_1, \cdots, x_n$ space, where R need not be simply connected. A curve $x_i = x_i(t)$, $i = 1, \cdots, n$, in R will be called *admissible* if it is of a type for which it has been proved that

$$\int P_1 dx_1 + P_2 dx_2 + \cdots + P_n dx_n \tag{1}$$

is independent of the path in any simply connected region in which

$$\frac{\partial P_i}{\partial x_j} = \frac{\partial P_j}{\partial x_i}, \quad i, j = 1, 2, \cdots, n ; \tag{2}$$

for example, rectifiable curves; or, curves such that $x_i(t)$ is continuous, $0 \leqq t \leqq 1$, and $x_i'(t)$ is "piecewise" continuous, with a finite number of finite jumps.

When not otherwise specified, the index i runs from 1 to n.

We shall say that curve

$$C_1 : x_i = f_i(t), \quad 0 \leqq t \leqq 1, \quad \text{with } f_i(0) = a_i, \ f_i(1) = b_i, \tag{3}$$

can be *deformed continuously with end points fixed* onto curve

$$C_2 : x_i = g_i(t), \quad 0 \leqq t \leqq 1, \quad \text{with } g_i(0) = a_i, \ g_i(1) = b_i, \tag{4}$$

in a region R, if there exists a continuous function $\mu(t)$, $0 \leqq t \leqq 1$, with $\mu(0) = 0$, $\mu(1) = 1$ and $0 \leqq \mu(t) \leqq 1$, and a set of functions $h_i(t, \alpha)$, continuous for

$$0 \leqq t \leqq 1, \quad 0 \leqq \alpha \leqq 1, \tag{5}$$

such that R contains the set H given by

$$x_i = h_i(t, \alpha), \quad (t, \alpha) \text{ on } (5), \tag{6}$$

and such that

$$h_i(0, \alpha) \equiv a_i, \quad h_i(1, \alpha) \equiv b_i, \tag{7}$$

(8) $$h_i(t, 0) \equiv f_i(t)$$

(9) $$h_i(t, 1) \equiv g_i[\mu(t)] .$$

The reason for taking a general function $\mu(t)$ rather than $\mu(t) \equiv 1$, is to permit C_1 to be mapped many-to-one on C_2.

THEOREM. *Let C_1 and C_2 be admissible curves given by* (3) *and* (4), *and suppose C_1 can be deformed continuously onto C_2 with end points fixed, in a region R. Let $P_1(x)$, $P_2(x)$, $\cdots$, $P_n(x)$ be of class C^1 and satisfy* (2) *in R. Then* (1) *has the same value when taken along C_1 from (a) to (b) as when taken along C_2 from (a) to (b).*

Proof of the theorem: We extend the definition of $h_i(t, \alpha)$ by defining, for $0 \leqq t \leqq 1$ and $1 \leqq \alpha \leqq 2$,

(10) $$h_i(t, \alpha) \equiv g_i\{\mu(t) + (\alpha - 1)[t - \mu(t)]\} .$$

We note that $h_i(t, 1)$ as given by (10) agrees with that given by (9), that $h_i(t, 2) \equiv g_i(t)$, that (7) remains valid, and that the quantity within the braces in (10) has values from 0 to 1 inclusive. It is now clear that, without loss of generality, we may assume that (9) can be replaced by

(11) $$h_i(t, 1) \equiv g_i(t) , \quad 0 \leqq t \leqq 1 .$$

Since H as defined in (6) is closed and R is open, there exists a constant $D > 0$ such that if $p \in H$ and the distance from p to $q \in E^n$ is less than D, then $q \in R$. Since the h_i are continuous, hence uniformly continuous on (5), there is an integer $m > 0$ such that if $|t - t'| < 1/m$ and $|\alpha - \alpha'| < 1/m$, with t, t', α, α' satisfying (5), then

(12) $$|h_i(t, \alpha) - h_i(t', \alpha')| < \frac{D}{2\sqrt{n}}$$

and such that furthermore, for $|t - t'| < 1/m$,

(13) $$|f_i(t) - f_i(t')| < \frac{D}{2\sqrt{n}} , \quad |g_i(t) - g_i(t')| < \frac{D}{2\sqrt{n}} .$$

Now consider the m^2 oriented squares $Q(r, s)$ in the t, α plane, $r, s = 0, 1, \cdots, m-1$, where $Q(r, s)$ has vertices, in cyclic order,

(14) $$(t, \alpha) = (\frac{r}{m}, \frac{s}{m}), (\frac{r+1}{m}, \frac{s}{m}), (\frac{r+1}{m}, \frac{s+1}{m}), (\frac{r}{m}, \frac{s+1}{m}) ,$$

and the m^2 oriented closed curves $K(r, s)$ in x space corresponding as follows. Let $X(i, j)$ be the point determined by (6) with $t = i/m$ and $\alpha = j/m$, $i, j = 0, 1, \cdots, m$. Then $K(r, s)$ consists of four oriented arcs joining the points $X(r, s)$, $X(r+1, s)$, $X(r+1, s+1)$, $X(r, s+1)$ in cyclic order, each arc being a straight segment except in the following three cases.

i) The segment is omitted when the two end points in x space are coincident, as, according to (6) and (7), is the case (with $r = m-1$) for $X(m, s)$, $X(m, s+1)$, and also (with $r = 0$) for $X(0, s+1)$, $X(0, s)$.

ii) Points $X(r, 0)$, $X(r+1, 0)$ are joined not by a line segment, but by the part of C_1 determined by $\frac{r}{m} \leqq t \leqq \frac{r+1}{m}$, as is possible in view of (3) and (8), with orientation that of increasing t.

iii) Points $X(r+1, m)$, $X(r, m)$ are joined not by a line segment, but by the part of C_2 determined by $\frac{r}{m} \leqq t \leqq \frac{r+1}{m}$, as is possible in view of (4) and (11), with orientation that of decreasing t.

If now we take $\int \Sigma P_i dx_i$ around the m^2 closed curves $K(r, s)$ determined as above, then, since the vertices of $K(r, s)$ are the images of those of $Q(r, s)$ under (6), we see by examination of (14) that all segments except those under cases ii) and iii) above are straight and are traversed once in each of two opposite senses (or possibly an equal number of times in two opposite senses, in cases of coincidences), and there remains, in view of ii), iii), (3) and (4), only the integral from *(a)* to *(b)* along C_1 plus the integral from *(b)* to *(a)* along C_2. Hence, if we can show that $\int \Sigma P_i dx_i = 0$ for each $K = K(r, s)$, the proof will be complete.

Now from (12), (14), the choice of m, and the formula for distance in E^n, we see that for each K the distances from any one vertex to the other three vertices (whether or not distinct) are all less than $D/2$. Since the four or fewer arcs of K determined by the vertices, are either straight segments in x space or are subject to (3) and (13) or to (4) and (13), with t on an interval of length $< 1/m$, we infer that all points of K are distant $< D$ from any one vertex. Since the interior of the sphere in E^n of radius D and center at a vertex of K is a simply connected region in R, we infer from (2) that $\int_k \Sigma P_i dx_i = 0$. Hence the theorem is true.

By easily made modifications of the proof above, the result can be extended to cover the case of two closed curves C_1 and C_2 such that C_1 can be deformed continuously into C_2 in the region R. In this case the hypotheses do not require any fixed points. We omit the details.

We observe that, in both cases, the intermediate positions of the deformations need not be admissible curves.

Queens College
Flushing, N. Y.

CURRENT PAPERS AND BOOKS

Edited by H. V. Craig

This department will present comments on papers previously published in the MATHEMATICS MAGAZINE, lists of new books, and book reviews.

In order that errors may be corrected, results extended, and interesting aspects further illuminated, comments on published papers in all departments are invited.

Communications intended for this department should be sent in duplicate to *H. V. Craig, Department of Applied Mathematics, University of Texas, Austin 12, Texas.*

COMMENT ON "RESISTANCE CIRCUITS AND THINGS SYNTHESIZED BY NUMBER THEORY"

Melvin Hochster

We have $\frac{1}{r_1}+\frac{1}{r_2}=\frac{1}{R}$. Then $(r_1-R)(r_2-R)=R^2$. Clearly, we obtain one solution for each factorization of R^2 (ab and ba are considered the same factorization; ab and $(-a)(-b)$, distinct). For positive solutions we restrict ourselves to positive factorizations. Then $N(R)=\frac{1}{2}d(R)+\frac{1}{2}$ (since R^2 is a square). For $R=120$ there are 32 solutions. Mr. Benson omitted, for example, $(r_1, r_2)=(192, 320)$, corresponding to $120^2=(72\cdot 200)$.

The mistake in Mr. Benson's reasoning is that he assumes that because X is even, $X=2\alpha\beta$ and $V=\alpha^2-\beta^2$. Actually, there may be (and always are) solutions in which $X=\alpha^2-\beta^2$ and $V=2\alpha\beta$ which are not obtainable in the other form.

A form of this problem appeared on Part I of this year's William Lowell Putnam Mathematical Competition, held December 3, 1960.

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TERMINAL DIGITS OF $MN(M^2-N^2)$ A CORRECTION AND AN EXTENSION

C. W. Trigg

Harry M. Gehman points out that the treatment of this topic in *Mathematics Magazine*, **34**, 159-60, January-February, 1961 is not correct for all values of M and N. He cites the contra-example $M=23$, $N=14$ which makes $p=6$ and *not* 4 as was given in the article when obtained from $m=3$, $n=4$.

The terminal (unit's) digit, p, of $P=MN(M^2-N^2)$ was treated as a

signless number. As such, the square array of the values of p and the subsequent treatment in the article does hold for digital values of M, N and for $M \geqq N$, $m \geqq n$ as well as for $M \leqq N$, $m \leqq n$. However for $M \geqq N$, $m \leqq n$ and $M \leqq N$, $m \geqq n$ another square array applies in which each element is the complement of the corresponding element in the first array. The recorded properties of the first array also apply to the second one.

$M \geqq N$, $m \geqq n$; $M \leqq N$, $m \leqq n$

$n \backslash m$	1	2	3	4	5	6	7	8	9
1	0	6	4	0	0	0	6	4	0
2	6	0	0	6	0	4	0	0	6
3	4	0	0	4	0	6	0	0	4
4	0	6	4	0	0	0	4	6	0
5	0	0	0	0	0	0	0	0	0
6	0	4	6	0	0	0	6	4	0
7	6	0	0	4	0	6	0	0	6
8	4	0	0	6	0	4	0	0	4
9	0	6	4	0	0	0	6	4	0

$M \geqq N$, $m \leqq n$; $M \leqq N$, $m \geqq n$

$n \backslash m$	1	2	3	4	5	6	7	8	9
1	0	4	6	0	0	0	4	6	0
2	4	0	0	4	0	6	0	0	4
3	6	0	0	6	0	4	0	0	6
4	0	4	6	0	0	0	6	4	0
5	0	0	0	0	0	0	0	0	0
6	0	6	4	0	0	0	4	6	0
7	4	0	0	6	0	4	0	0	4
8	6	0	0	4	0	6	0	0	6
9	0	4	6	0	0	0	4	6	0

It will be observed that one array goes into the other by rotation about the main cross-diagonal (lower left to upper right).

The terminal digits may be rearranged into two new arrays, by interchanging the lower-left triangles to give one array true for $M \geqq N$ and the other for $M \leqq N$:

$M \geqq N$

$n \backslash m$	1	2	3	4	5	6	7	8	9
1	0	6	4	0	0	0	6	4	0
2	4	0	0	6	0	4	0	0	6
3	6	0	0	4	0	6	0	0	4
4	0	4	6	0	0	0	4	6	0
5	0	0	0	0	0	0	0	0	0
6	0	6	4	0	0	0	6	4	0
7	4	0	0	6	0	4	0	0	6
8	6	0	0	4	0	6	0	0	4
9	0	4	6	0	0	0	4	6	0

$M \leqq N$

$n \backslash m$	1	2	3	4	5	6	7	8	9
1	0	4	6	0	0	0	4	6	0
2	6	0	0	4	0	6	0	0	4
3	4	0	0	6	0	4	0	0	6
4	0	6	4	0	0	0	6	4	0
5	0	0	0	0	0	0	0	0	0
6	0	4	6	0	0	0	4	6	0
7	6	0	0	4	0	6	0	0	4
8	4	0	0	6	0	4	0	0	6
9	0	6	4	0	0	0	6	4	0

Here again are two arrays which go into each other by rotation about the main cross-diagonals. Each of these arrays, which are mirror images

of each other, has the following properties :

1) Non-zero elements symmetrical to the principal diagonal are complementary, as are those symmetrical to the main cross-diagonal.

2) Non-zero elements symmetrical to the bisectors of the sides are complementary.

3) $p_{m,n} = p_{m+5,n} = p_{m+5,n+5} = p_{m,n+5}$. Thus vertices of squares having six elements on a side are equal. There are 16 such squares.

4) The array is symmetrical to the central element, $p_{5,5}$.

5) The line of zeros bisecting the sides divides the array into four congruent square arrays. Each of these sub-arrays has diagonals composed of zeros. Otherwise, each sub-array has a central square of zeros surrounded by an octagon in which 6's and 4's alternate, so that the elements on each side of the octagons are complementary.

6) The value, D, of each sub-array considered as a fourth order determinant is -400. Thus $D/$(sum of its elements) is -10.

7) The square with corners at the centers of the four sides itself has congruent 5-element sides in which, while the digits 4 and 6 alternate they are separated by zeros. Nested in this are a 4-element square with perimeter similar to that of the sub-array, and 3, 2, and 1-element squares composed of zeros.

8) The sum of the elements in each row (or column) other than the fifth (middle) one is 20.

9) The 4's may be traversed completely by Knight's moves, as may the 6's. The joins of the elements of each closed path form an expanded swastika. The two swastikas in each array are mirror images.

Los Angeles City College.

BOOK REVIEWS

Algebra Problems. By Donald S. Russell. Barnes and Noble, Inc., New New York, 1960, VII and 134 pp., $1.75.

For those teachers who feel that students should see a large number of problems worked out in detail, this paperback book will have some appeal.

The book briefly lists the rules of algebra and works out a number of problems illustrating each rule. It is not intended as a text but as a supplement to the teacher and the text. It contains solved problems of almost every type usually encountered in a standard course in intermediate algebra.

Robert E. Horton

The Higher Arithmetic. By H. Davenport. Harper, N.Y., 1960, 168 pp. $1.35.
The Skeleton Key of Mathematics. By D. E. Littlewood. Harper, N. Y., 1960, 136 pp. $1.25.

These are two of the excellent series of Harper Torchbooks' Science Library paperbound reprints. The typography, paper and binding are better than those found in most paperbacks.

Davenport's book, first published in 1952, is a clearly written introduction to the theory of numbers. Covering elementary aspects of such topics as congruences, continued fractions and quadratic forms, the book lacks only problems for the student to make it an ideal text for the beginner.

A Simple Account of Complex Algebraic Theories is the sub-title of Littlewood's book, first published in 1949. The author's ambitious goal is to describe for the general intelligent reader the main ideas underlying the theories of congruences, ideals, matrices, tensors, groups, and other areas of algebra. Often the discussion is difficult to follow, but some of the topics are not simple. This is not an easy book, but is a book much more valuable than its price implies.

J. M. C. Hamilton

BOOKS RECEIVED FOR REVIEW

Theory of Functions of a Real Variable. By Edwin Hewitt. Holt, Rinehart, and Winston, Inc., New York, 1960, 326 pages. $4.00 (paper bound).

Solution of Equations and Systems of Equations. By A. M. Ostrowski. Academic Press, New York and London, 1960, ix + 200 pages. $6.80.

Senior Technical Mathematics. Edited by A. H. Heywood. The Macmillan Company of Canada, Limited, Toronto, 1960, xii + 558 pages. $4.50.

Mathematics in Practice (Revised). By A. E. Brown, E. D. Bridge, and W. J. Morrison. The Macmillan Company of Canada, Limited, Toronto, 1960, xi + 418 pages. $3.10.

PROBLEMS AND QUESTIONS

Edited by

Robert E. Horton

Readers of this department are invited to submit for solution problems believed to be new and subject matter questions that may arise in study, in research, or in extra-academic situations. Proposals should be accompanied by solutions, when available, and by any information that will assist the editor. Ordinarily, problems in well-known textbooks should not be submitted.

Solutions should be submitted on separate, signed sheets. Figures should be drawn in India Ink and exactly the size desired for reproduction.

Send all communications for this department to *Robert E. Horton, Los Angeles City College, 855 North Vermont Ave., Los Angeles 29, California.*

PROPOSALS

439. *Proposed by Maxey Brooke, Sweeney, Texas.*

A fly is resting on the floor on the north side of the base of a circular column. It suddenly realizes that a spider is resting diametrically opposite him. The fly starts crawling due north. At the same time the spider starts traveling due east. As everyone knows, a spider can crawl three times as fast as a fly.

When the fly has crawled 3 inches, he sees the spider just emerging from behind the curve of the column. He realizes that all is lost and freezes on the spot. The spider turns and crawls in a straight line to the fly and devours him. How far does the spider travel?

440. *Proposed by Huseyin Demir, Kandilli, Eregli, Kdz., Turkey.*

Consider a packing of circles of radius r such that each is tangent to its six surrounding circles. Let a larger circle of radius R be drawn concentric with one of the small circles. If N is the number of small circles contained in the larger circle, prove that

$$N = 1 + 6n + 6\sum_{p=1}^{n} [\tfrac{1}{2}(\sqrt{4n^2 - 3p^2} - p)]$$

where $n = [\tfrac{1}{2}(\frac{R}{r} - 1)]$, the square brackets designating the greatest integer function.

441. *Proposed by Vladimir F. Ivanoff, San Carlos, California.*

A skew quadrilateral $ABCD$ lies entirely on a ruled quadric surface. Show that for any point $P(x, y, z)$ on the quadric,

$$\frac{PA \cdot PC \sin\beta \sin\delta}{PB \cdot PD \sin\alpha \sin\gamma} = \lambda$$

where α, β, γ, and δ are the dihedral angles whose edges are PA, PB, PC, and PD respectively and λ is a constant.

442. *Proposed by C. W. Trigg, Los Angeles City College.*

A 10-gram mass and a 120-gram mass are connected by a strong, light 100 cm. thread. They are placed on the horizontal top of a table 100 cm. high, with the 10-gram mass just over an edge to which the taut string is then perpendicular. The system is then released. Where will the heavier mass strike the floor? (The coefficient of friction with the table top is 0.04).

443. *Proposed by B. Weesakul, The University of Western Australia.*

Show that the characteristic roots of the $n \times n$ matrix

$$\begin{pmatrix} 0 & p & 0 & 0 & 0 & 0 \\ q & 0 & p & 0 & \cdots & 0 \\ 0 & q & 0 & p & \cdots & 0 \\ 0 & & & & & \\ \vdots & & . & q & 0 & p \\ 0 & 0 & \cdots & 0 & q & 0 \end{pmatrix}$$

are $S_v = 2(pq)^{\frac{1}{2}} \cos \frac{v\pi}{n+1}$, $p > 0$, $q > 0$ and $v = 1, 2, \cdots, n$.

444. *Proposed by Melvin Hochster, New York.*

If k and n are integers such that $0 \leq k \leq n$, $[p]$ represents the greatest integer part of p, and $\binom{q}{-1} = 0$, prove that

$$\sum_{i=0}^{k} \left\{ \binom{k}{[\frac{1}{2}k]-i} - \binom{k}{[\frac{1}{2}k]-i-1} \right\} \left\{ \binom{2n-k}{[n-\frac{1}{2}k]-i} - \binom{2n-k}{[n-\frac{1}{2}k]-i-1} \right\} = \binom{2n}{n} - \binom{2n}{n-1}$$

445. *Proposed by the late Victor Thebault, Tennie, Sarthe, France.*

Given in a plane a circle (O) on which one takes two points A and B, and a point M of which the power with respect to (O) is k. Prove that the polar of M with respect to (O) meets AB in a point P such that

$$\frac{PA}{PB} = \frac{\overline{MA}^2 - k}{\overline{MB}^2 - k}.$$

SOLUTIONS

The Pharaoh's Windfall

418. [September 1960] *Proposed by Maxey Brooke, Sweeney, Texas.*

The Pharaoh of Egypt, falling on hard times, decided to replenish the royal treasury by selling the Great Pyramid at one obol per cubic yard. The Emperor of Ethiopia, who had always wanted a pyramid, sent his royal surveyor to check on the measurements.

Now a yard was the length of the ruler's arm, and the Emperor had a longer arm than did the Pharaoh, so the volume of the pyramid as determined by the Ethiopian and Egyptian surveyors differed. There resulted much diplomatic wrangling over price.

It was finally decided to submit the question to the King of Babylon and abide by his decision. The King looked into the question and found that the Emperor's arm was as much longer than his own as the Pharaoh's was shorter. He reasoned that since the Babylonian yard was the average of the Ethiopian and Egyptian yards, the volume of the pyramid using his measure would be the average of the volumes as determined by the Egyptian and Ethiopian surveyors.

It is reputed that the Pharaoh came into an inheritance and no longer needed money so the pyramid was never sold.

Nevertheless, which ruler did the King's decision favor?

Solution by Harvey Walden, Rensselaer Polytechnic Institute.

If we represent the length of the Babylonian yard as L, we can write $(L+d)$ as the length of the Ethiopian yard and $(L-d)$ as the length of the Egyptian yard, where d is a positive increment. Thus, a cubic yard as measured by Babylonians would be L^3 cubic units, while the Ethiopian version would be $(L+d)^3$ or $(L^3+3L^2d+3Ld^2+d^3)$ cubic units and the Egyptian version $(L-d)^3$ or $(L^3-3L^2d+3Ld^2-d^3)$ cubic units.

Now the actual average of the Ethiopian and Egyptian yards would be one-half the sum of the cubic measurements, i. e., $\frac{1}{2}[(L+d)^3+(L-d)^3]$, *not* one-half the sum of the linear measurements, i. e., $\frac{1}{2}[L+d)+(L-d)]=L$. This actual average, which works out to be (L^3+3Ld^2), represents the average of the Ethiopian and Egyptian cubic yards, just as L represents the average of the two linear yards. Since $(L^3+3Ld^2)>L^3$, where L^3 is the miscomputed average cubic yard, any volume, including that of the Great Pyramid, measured by the L^3 standard would turn out to contain more cubic yards than if measured by the (L^3+3Ld^2) standard. Thus, had the Pharaoh of Egypt accepted the faulty reasoning of the King of Babylon, he would have overcharged the Emperor of Ethiopia quite a few obols for the pyramid.

Also solved by D. A. Breault, Sylvania Electronics Systems, Watham, Massachusetts; Sam Kravitz, East Cleveland, Ohio; Sidney Kravitz, Dover, New Jersey; and the proposer. One incorrect solution was received.

Constant Speed Curve

419. [September 1960] *Proposed by Huseyin Demir, Kandilli, Eregli, Kdz., Turkey.*

Determine the path in a vertical plane such that when a particle moved, under gravity, with an initial velocity v_0 from a point of the path, the particle maintained a constant speed along the path. Assume no friction.

Solution by the proposer.

Let Ox, Oy be the axes of coordinates taken in the vertical plane such

that Oy points downward and Ox to the left. Let the particle be dropped from O. It reaches the velocity v_0 at a point A of Oy with $y_0 = OA = \dfrac{v_0^2}{2g}$.

Since there is no friction, the velocity along the path is the projection of the velocity $v = \sqrt{2gy}$, and we write $v_0 = v \cos \alpha$ where

$$v_0 = \sqrt{2gy_0}\,, \quad v = \sqrt{2gy}\,, \quad \cos^2\alpha = \frac{1}{(1+tg^2\alpha)} = \frac{1}{(1+y'^2)}$$

and get $y_0 = y/(1+y'^2)$.

The variables separate and give

$$x = \int_{y_0}^{y} \frac{dy}{\sqrt{(y-y_0)/y_0}} = ½\sqrt{y_0}\sqrt{y-y_0}$$

$$y = \frac{4x^2}{y_0} + y_0$$

$$y = \frac{8g}{v_0^2}x^2 + \frac{v_0^2}{2g}\,.$$

The path is a parabola tangent to Oy at A, Oy being the tangent at the vertex.

A Factorial Congruence

420. [September 1960] *Proposed by Leonard Carlitz, Duke University.*

Let

$$Q_{r,s} = \frac{(rs)!}{r!\,(s!)^r}\,.$$

Show that

$$Q_{r,ps} \equiv Q_{r,s} \pmod p$$

where p is a prime.

Solution by the proposer.

We have

$$Q_{r,s} = \prod_{j=1}^{r} \binom{js-1}{s-1},$$

so that

(1)
$$Q_{r,ps} = \prod_{j=1}^{r} \binom{jps-1}{ps-1}\,.$$

Now it follows from

$$(1+x)^{jps-1} \equiv (1+x^p)^{js-1}(1+x)^{p-1} \pmod p$$

that

$$\binom{jps-1}{ps-1} \equiv \binom{js-1}{s-1} \pmod{p} .$$

Thus (1) becomes

$$Q_{r,ps} \equiv \prod_{j=1}^{r} \binom{js-1}{s-1} \equiv Q_{r,s} \pmod{p} .$$

A Circle of Apollonius

421. [September 1960] *Proposed by James Churchyard, Avondale, Arizona.*

Given a straight line segment ABC such that $AB \neq BC$. What is the locus of points P such that angle APB is equal to angle BPC?

I. *Solution by Joseph D. E. Konhauser, State College, Pennsylvania.*

By a well-known theorem of Euclid (Elements VI, Prop. 3), $AP/PC = AB/BC$ = constant $\neq 1$. Hence the locus of P is a circle of Apollonius.

II. *Solution by Sister M. Stephanie, Georgian Court College, Lakewood, New Jersey.*

Place the points on the x-axis of a Cartesian plane so that $A = (-a, 0)$, $B = (0, 0)$, $C = (c, 0)$ and $a \neq c$. Let $P = (x, y)$. If $\sphericalangle APB = \sphericalangle BPC$ then $\tan APB = \tan BPC$. The slopes of lines AP, BP and CP are all known. Using the formula for the tangent of an angle formed by two lines whose slopes are known, we find the locus to be $x^2 - \dfrac{2ac}{a-c}x + y^2 = 0$, a circle with its center on the x-axis and radius equal to $\left|\dfrac{ac}{a-c}\right|$. As a approaches c the center moves to infinity on the x-axis, and the circle degenerates into the y-axis.

III. *Comments by C. N. Mills, Sioux Falls College, South Dakota.*

By analytical geometry it is not difficult to prove that the required locus is a circle that passes through the point B, and has its center on the line segment extended. The circle cuts the extended line at point K. The points A and C are inverse points with respect to the locus as the circle of inversion. The points A, B, C, and K form a harmonic range. The points B and K are the intersections of the internal and external bisectors of the angle P with the base line.

Also solved by Huseyin Demir, Kandilli, Eregli, Kdz., Turkey; William Kantor, Brooklyn College; C. F. Pinzka, University of Cincinnati; Paul D. Thomas, Coast and Geodetic Survey, Washington, D. C.; C. W. Trigg, Los Angeles City College, and Harvey Walden, Rensselaer Polytechnic Institute.

A Differential Equation

422. [September 1960] *Proposed by M. S. Klamkin, AVCO, Wilmington, Massachusetts.*

Solve the differential equation

$$\{x(1-\lambda)D^2+(x\phi'+1)D+x\phi''+\phi'\}y=0\ ,$$

where λ is a constant and ϕ is a given function of x.

Solution by P. D. Thomas, Coast and Geodetic Survey, Washington, D.C.

Using primes to denote differentiation of y with respect to x, rearrange and collect the terms of the given differential equation to get

$$(xy\phi')'+(1-\lambda)(xy')'+\lambda y'=0\ ,$$

a first integral being at once

$$xy\phi'+(1-\lambda)xy'+\lambda y=C \quad \text{(constant)}\ ,$$

or

$$\text{(1)} \qquad y'+\frac{y(x\phi'+\lambda)}{(1-\lambda)x}=\frac{C}{(1-\lambda)x}\ .$$

Now (1) is linear and the known solution is

$$\text{(2)} \qquad y=e^{-\int P\,dx}\left(\int Qe^{\int P\,dx}dx+k\right)$$

where from (1),

$$P=\frac{x\phi'+\lambda}{(1-\lambda)x}, \quad Q=\frac{C}{(1-\lambda)x}$$

and k is a constant.

$$\text{(3)} \qquad \int P\,dx=\frac{1}{1-\lambda}\int\left(\phi'+\frac{\lambda}{x}\right)dx=\frac{\phi+\lambda\ln x}{1-\lambda}.$$

$$\text{(4)} \qquad \int Qe^{\int P\,dx}dx=\frac{C}{1-\lambda}\int e^{\phi/(1-\lambda)}x^{(2\lambda-1)/(1-\lambda)}dx\ .$$

The solution may then be written from (2), (3), and (4) as

$$y=x^{\lambda/\lambda-1)}e^{-\phi/(1-\lambda)}\left[\frac{CI}{1-\lambda}+k\right] \quad \text{where} \quad I=\int e^{\phi/(1-\lambda)}x^{(2\lambda-1)/(1-\lambda)}dx\ .$$

Also solved by I. Dale Ruggles, San Jose State College; Ilija Sapkarev, University of Skopje, Yugoslavia; and the proposer.

A Difference Equation

424. [September 1960] *Proposed by Edward T. Frankel, U. S. Department of Health, Education and Welfare.*

The general term of a polynomial series is

$$u_r=\binom{a+dr}{n}.$$

Show that $\Delta^n u_0=d^n$.

Solution by Dmitri Thoro, San Jose State College.
If

$$u_r = \binom{a+dr}{n} \quad \text{then} \quad \Delta^n u_0 = d^n .$$

Proof. u_r is a polynomial in r (of degree n) with leading coefficient $d^n/n!$. Thus from the Calculus of Finite Differences (e.g., Richardson, *An Introduction to the Calculus of Finite Differences*, pp. 8-9)

$$\Delta^n u_0 = \frac{d^n}{n!} \cdot h! \Big]_{r=0} = d^n .$$

Also solved by R. V. Parker, Bressingham, Diss, Norfolk, England; and the proposer.

Comment on Problem 395

395. [November 1959 and May 1960] *Proposed by Sidney Kravitz, Dover, New Jersey.*

It is well known that $f(n) = n^2 - n + 41$ yields prime numbers for $n \leqq 40$. Show that $f(n)$ contains at most two prime factors for $n \leqq 420$.

Comment by William E. Christilles, San Antonio, Texas.

In the 1960 May-June issue of the Mathematics Magazine there appeared a solution to Problem 395 which states that $f(n) = n^2 - n + 41$ is the product of at most two prime factors for $n \leqq 420$. To prove this, the statement was made that: "$f(n) = n^2 - n + 41$ is a prime for all values of n unless $n = 41k$ or $n - 1 = 41k$." If this were the case, the function $f(n) = n^2 - n + 41$ would be a prime generator, excluding $n = 41k$ and $n - 1 = 41k$. That the statement is false may be seen immediately by an example. Let $n = 50$. $50 \neq 41k$ and $50 - 1 \neq 41k$. But $f(50) = (50)^2 - (50) + 41 = 2491 = (47)(53)$.

Perhaps a simple approach to this problem, without going into tedious calculations, would be as follows: The form $x^2 + xy + 41y^2$, in integers x and y, of which the form $n^2 - n + 41$ is a sub-set, is known to have division closure; moreover 41 is the least prime (or integer for that matter) which can be represented by either set. Now by considering all possible products of three prime factors p_i, p_u, and p_w, from the set $x^2 + xy + 41y^2$ such that $p_i p_u p_w \leqq (420)^2 - (420) + 41$, and showing that $p_i p_u p_w \neq n^2 - n + 41$, for any $n \leqq 420$, we would prove the statement in question. Since 41 is the least possible prime which can be used, this would not require excessive labor. It is perhaps worthy of mention that although $(41)^3$, for example, may not belong to the set $n^2 - n + 41$, it must belong to the set $x^2 + xy + 41y^2$ due to the property of closure for multiplication.

QUICKIES

From time to time this department will publish problems which may be solved by laborious methods, but which with the proper insight may be disposed of with dispatch. Readers are urged to submit their favorite problems of this type, together with the elegant solution and the source, if known.

Q 275. Show that the moments of inertia about all centroidal axes of an area with n-fold ($n \geqq 3$) symmetry are the same. [*Submitted by M. S. Klamkin.*]

Q 276. Show that $x^m D^{m+n} x^n \equiv D^n x^{m+n} D^m$ where D is the differential operator $\dfrac{d}{dx}$. [*Submitted by M. S. Klamkin.*]

Q 277. Determine an infinite sequence of integers for which the finite differences of all orders are the same as the original sequence. [*Submitted by Brother Alfred.*]

Q 278. During a period of days, it was observed that when it rained in the afternoon, it had been clear in the morning, and when it rained in the morning, it was clear in the afternoon. It rained on 9 days, and was clear on 6 afternoons and 7 mornings. How long was this period? [*Submitted by C. W. Trigg.*]

Q 279. For positive integral $n > 1$ show that $\frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{n^2} > 1$. [*Submitted by Barney Bissinger.*]

Q 280. The following letter was received by the Editor:

Baraboo, Wisconsin
January 13, 1961

Dear Problem Solvers:

Determine the terms of a Fibonacci series whose first, second, and thirteenth terms are right now.

Yours very truly,

James H. Hill, Jr.

Answers are on page 248.

NAMES OF MATHEMATICIANS IN TIMELY ANAGRAMS

C. W. Trigg

1) A CALM RUIN
2) AND RAG
3) MA, JUAN RAN
4) ARM ONCE
5) I'M THERE
6) I POLL NO U.S.A.
7) PAR ON ICE
8) NO HARPS
9) AIM HIS CLUB
10) A HILL TOP

Unscrambled names are on page 247.

A NOTE ON THE PROBABILISTIC INEQUALITIES*

H. P. Kuang

Let the discrete marginal distributions of the two stochastic variables X and Y be $F_1 = \{p_j\}$ and $F_2 = \{q_j\}$ $(j = 1, \cdots, r)$ respectively, where $p_j = P(X = I_j)$, $q_j = P(Y = I_j)$; and I_j is an arbitrary interval containing a finite number of the mass points. Let further the corresponding observations of X and Y be $\{n_j\}$ and $\{m_j\}$ respectively, where the sample values n_j and m_j belong to the interval I_j and $\sum_{j=1}^{r} n_j = n$, $\sum_{j=1}^{r} m_j = m$. Let F_n and F_m be the empirical distributions of X and Y respectively, and let θ be any positive number. If $F_1 = F_2$, then

$$P\left\{\phi(F_n, F_m) \geqq 1 - \frac{\theta^2}{2}\right\} > 1 - \frac{r^2 + r - 1}{\theta^4}\left(\frac{1}{\sqrt{n}} + \frac{1}{\sqrt{m}}\right)^4$$

where

$$\phi(F_n, F_m) = \sum_{j=1}^{r} \sqrt{\frac{n_j}{n}\frac{m_j}{m}}\ .$$

Proof:

Let us define the metric

$$M^2(F_n, F_m) = \sum_{j=1}^{r}\left(\sqrt{\frac{n_j}{n}} - \sqrt{\frac{m_j}{m}}\right)^2 .$$

Then

$$M(F_n, F_m) \leqq M(F_1, F_n) + M(F_2, F_m)\ .$$

Hence, by the Minkowski inequality, we have

$$\begin{aligned} P\{M(F_n, F_m) \leqq \theta\} &\geqq P\{M(F_1, F_n) + M(F_2, F_m) \leqq \theta\} \\ &\geqq 1 - \frac{E\{M(F_1, F_n) + M(F_2, F_m)\}^4}{\theta^4} \\ &\geqq 1 - \frac{\{[E\overline{M(F_1, F_n)}^4]^{1/4} + [E\overline{M(F_2, F_m)}^4]^{1/4}\}^4}{\theta^4}\ . \end{aligned}$$

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*An invited paper presented at the Southeastern Sectional Meeting of the Mathematical Association of America, held at the University of South Carolina, April 1-2, 1960.

Since

$$E\overline{M(F_1, F_n)}^4 < \frac{r^2 + r - 1}{n^2},$$

consequently

$$P\{M(F_n, F_m) \leqq \theta\} > 1 - \frac{\left\{\left[\frac{r^2 + r - 1}{n^2}\right]^{1/4} + \left[\frac{r^2 + r - q}{m^2}\right]^{1/4}\right\}^4}{\theta^4}.$$

Now

$$M^2(F_n, F_m) = 2[1 - \phi(F_n, F_m)] .$$

Therefore

$$P\left\{\phi(F_n, F_m) \geqq 1 - \frac{\theta^2}{2}\right\} > 1 - \frac{r^2 + r - 1}{\theta^4}\left(\frac{1}{\sqrt{n}} + \frac{1}{\sqrt{m}}\right)^4 .$$

By the same argument, it can easily be shown that if $F_1 \neq F_2$, say $M(F_1, F_2) \geqq \alpha$, where $\alpha > \theta$, then

$$P\left\{\phi(F_n, F_m) < 1 - \frac{\theta^2}{2}\right\} > 1 - \frac{r^2 + r - 1}{(\alpha - \theta)^4}\left(\frac{1}{\sqrt{n}} + \frac{1}{\sqrt{m}}\right)^4 .$$

North Dakota State University

UNSCRAMBLED NAMES OF MATHEMATICIANS WITH MONTHLY ACROSTIC

1) MACLAURIN
2) ARGAND
3) RAMANUJAN
4) CREMONA
5) HERMITE
6) APOLLONIUS
7) POINCARE
8) RAPHSON
9) IAMBLICHUS
10) L'HOPITAL

EXCEPTIONAL CASE

Is our Max too free with brandy?
Overfond of applejacks?
Is the maximum of minims
Less than minimum for Max?

ANSWERS *(To Quickies on pages 243-244).*

A 275. The ellipse of inertia must be circular since three diameters of a proper ellipse cannot all be equal.

A 276. Since $D^n x = \Sigma a_r x^r D^r$, $x^n D^n = xD(xD-1)\cdots(xD-n+1)$. Consequently $x^m D^m$ and $D^n x^n$ commute.

A 277. One solution is a sequence of integers in geometric progression with constant ratio 2:

Sequence a, $2a$, $4a$, $8a$, $16a$, $\cdots$. First difference a, $2a$, $4a$, $8a$, $\cdots$.

Another solution is the Fibonacci sequence:

Sequence 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, $\cdots$.
First difference 0, 1, 1, 2, 3, 5, 8, 13, 21, $\cdots$.

A 278. There must have been $\frac{1}{2}(6+7-9)$ or 2 completely clear days, so there were $9+2$ or 11 days in the period.

A 279. Since each of the n^2-n terms which follow the first term is greater than $1/n^2$, it follows that

$$\frac{1}{n}+\frac{1}{n+1}+\cdots+\frac{1}{n^2}>\frac{1}{n}+\frac{n^2-n}{n^2}=\frac{1}{n}+1-\frac{1}{n}=1\,.$$

A 280. The proposer of this Quicky, Mr. James H. Hill, Jr., Miller Building, Baraboo, Wisconsin, has offered to send a gift package of Wisconsin cheese to the first two solvers who mail a correct solution directly to him. Get your solutions in the mail early! The solution to **Q 280** will appear in the next issue of the *Mathematics Magazine*.

Comments on Q 270

Leon Bankoff, R. Bumcrot, Arthur B. Brown, and M. S. Klamkin pointed out a shorter solution to **Q 270**.

$$a^2+b^2+c^2-ab-bc-ca=\tfrac{1}{2}[(a-b)^2+(b-c)^2-(c-a)^2]\,.$$

Klamkin pointed out that **Q 270** was the same as **Q 110.**

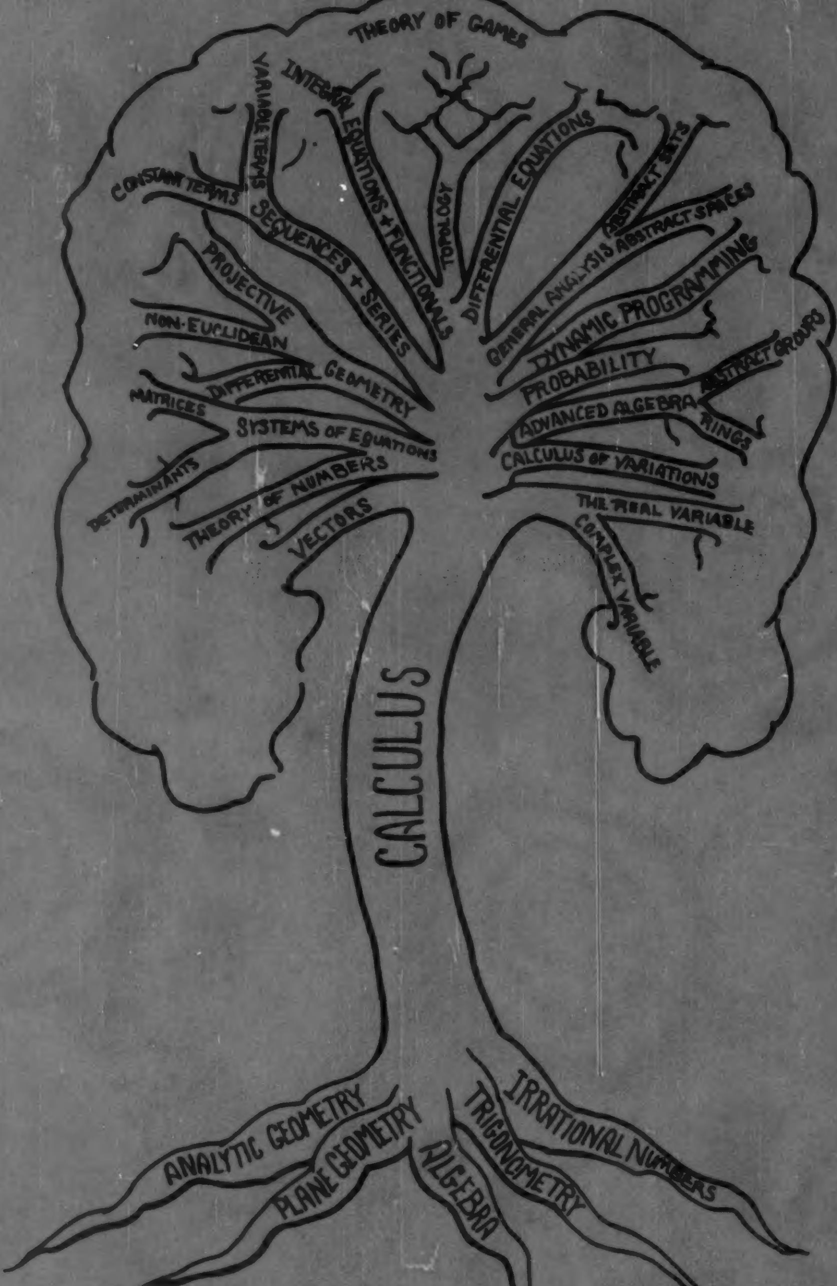
Comment on T 40

M. S. Klamkin wrote that in **T 40**, the next three digits could be anything. One can fit any sequence (even if an arbitrarily large number of terms are given) to make the next r (arbitrary) terms anything one desires. (See "On a Paradox," Pi Mu Epsilon Journal, Spring 1958.) To illustrate this, sometimes the sequence 42, 34, 28, 23, 14, 8, is given. Two possible next terms are Prince and Canal. These are local BMT subway stops in Manhattan.

TO A LINE

Though smartly free from garniture
Your widthless starving-slim
Abandonment of curvature
Insures for your progeniture,
And you, a future dim.

If you cannot remain content
With just one point ideal,
Then you, unswerving and unbent,
Will soon discover that you went
And made yourself unreal.



See inside back cover.